Interest-Rate Models: Course Notes

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Spot-Rate Models

- **Normal Rate (Gaussian) Models**
  - **Vasicek (1977)**
  - **Hull and White (1994)**
- **Lognormal Models**
  - **Black and Karasinski (1991) (BK)**
  - **Peterson, Stapleton and Subrahmanyam (2003), 2-factor BK**
- **Spot-rate Models**
  - **Assume a process for the spot short rate**
  - **Derive bond prices, given the spot rate process**
  - **Can be used to price derivatives (caps, swaptions)**

References

- **Vasicek, O., (1977), "An Equilibrium Characterization of the Term Structure,"** *Journal of Financial Economics*, **5, 177–188.**
- **Hull, J., and A. White (1994), "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models,"** *Journal of Derivatives*, **2, 37–48.**
- **Black, F., and P. Karasinski (1991), "Bond and Option Pricing when Short Rates are Lognormal,"** *Financial Analysts Journal*, **47, 52–59**
- **Peterson,S., Stapleton, R.C. and M. Subrahmanyam (2003), "A Multi-Factor Spot-Rate Model for the Pricing of Interest-rate Derivatives",** *Journal of Financial and Quantitative Analysis*

Lecture 1: The Gaussian Model

- **Vasicek-Hull and White type model**
- **Assume that the short (δ- period) rate follows a normal distribution, mean-reverting process**
- **Discrete time process**
- **Under risk-neutral measure**

Geometric Progressions

1.

$$S_1 = 1 + (1-k) + (1-k)^2 + ... + (1-k)^{n-1}$$

$$S_1 = \frac{1-(1-k)^n}{k}$$

2.

$$S_2 = 1 + (1-k)^2 + (1-k)^4 + ... + (1-k)^{2(n-1)}$$

$$S_2 = \frac{1-(1-k)^{2n}}{1-(1-k)^2}$$

Geometric Progressions

3.

$$S_3 = n+(n-1)(1-k)+(n-2)(1-k)^2+\dots$$
$$+2(1-k)^{(n-2)} + (1-k)^{(n-1)}$$

$$S_3 = \frac{1}{k}\left\{n-(1-k)\left[\frac{1-(1-k)^n}{k}\right]\right\}$$

4.

$$S_4 = n+(n-1)(1-k)^2+(n-2)(1-k)^4+\dots$$
$$+2(1-k)^{2(n-2)} + (1-k)^{2(n-1)}$$

$$S_4 = \frac{n-(1-k)^2\left[\frac{1-(1-k)^{2n}}{1-(1-k)^2}\right]}{1-(1-k)^2}$$

Short-Rate Model

$$r_{t+1} - r_t = k(a - r_t) + \varepsilon_{t+1}$$

short rate of interest, r_t

long-term mean of short rate, a

rate of mean reversion, k**,** $0 < k < 1$

ε_{t+1}**, drawing from a normal distribution,** $E_t(\varepsilon_{t+1}) = 0$**,** $var_t(\varepsilon_{t+1}) = \sigma^2$**.**

Hence,

$$r_{t+1} = ka + (1 - k)r_t + \varepsilon_{t+1} \quad (1)$$

Equation (1) holds for all t

Short-Rate Model

Example: Short rate:3-month Libor

$t = 0$

short rate, $r_0 = 0.05$

long term mean, $a = 0.06$

mean reversion, $k = 0.25$

variance, $\sigma^2 = 0.0004$

$$r_1 = 0.25(0.06) + 0.75(0.05) + \varepsilon_1$$

Mean and Variance of the Short-Rate

$$r_{t+1} = ka + (1-k)r_t + \varepsilon_{t+1} \quad (1)$$

$$r_{t+2} = ka + (1-k)r_{t+1} + \varepsilon_{t+2} \quad (2)$$

Substitute (1) in (2)

$$r_{t+2} = ka + (1-k)ka + (1-k)^2 r_t + (1-k)\varepsilon_{t+1} + \varepsilon_{t+2}$$

Mean of r_{t+2}:

$$E_t(r_{t+2}) = ka + (1-k)ka + (1-k)^2 r_t \quad (3)$$

Variance of r_{t+2}:

$$var_t(r_{t+2}) = (1-k)^2 var(\varepsilon_{t+1}) + var(\varepsilon_{t+2}) \quad (4)$$

Exercise: Find the mean and variance of r_{t+3}

Mean and Variance of r_{t+n}

$$\begin{aligned} r_{t+n} &= ka + (1-k)ka + (1-k)^2 ka + ... + (1-k)^{n-1} ka \\ &+ (1-k)^n r_t \\ &+ (1-k)^{n-1}\varepsilon_{t+1} + (1-k)^{n-2}\varepsilon_{t+2} + ... + \varepsilon_{t+n} \end{aligned}$$

Hence

$$\begin{aligned} E_t(r_{t+n}) &= ka + (1-k)ka + (1-k)^2 ka + ... \\ &+ (1-k)^{n-1} ka + (1-k)^n r_t \\ &= ka[1 + (1-k) + (1-k)^2 + ... + (1-k)^{n-1} \end{aligned}$$

Variance of r_{t+n}:

$$\begin{aligned} var_t(r_{t+n}) &= (1-k)^{2(n-1)} var(\varepsilon_{t+1}) \\ &+ (1-k)^{2(n-2)} var(\varepsilon_{t+2}) \\ &+ ... + var(\varepsilon_{t+n}) \end{aligned}$$

and if

$$var(\varepsilon_{t+1}) = var(\varepsilon_{t+2}) = ... = var(\varepsilon_{t+n}) = \sigma^2$$

$$var_t(r_{t+n}) = \sigma^2[(1-k)^{2(n-1)} + (1-k)^{2(n-2)} + ... + 1]$$

Mean and Variance of r_{t+n}

Using the results from Geometric Progression:

Mean of r_{t+n}:

$$E_t(r_{t+n}) = a[1 - (1-k)^n] + (1-k)^n r_t$$

Variance of r_{t+n}:

$$var_t(r_{t+n}) = \sigma^2 \left[\frac{1-(1-k)^{2n}}{1-(1-k)^2} \right]$$

Annualised st.dev.

$$std_t(r_{t+n}) = \sqrt{var_t(r_{t+n})/n}$$

Long-Bond Prices and Yields

Some key results:

1. **Mean of a lognormal variable:**

   Let x be normally distributed, with (μ, σ^2) then

$$E(e^{bx}) = e^{b\mu + 0.5(b^2\sigma^2)}$$

2. **Forward price of a bond:**

   The forward price of a 1-period bond is

$$B_{t,t+\tau,t+\tau+1} = E_t[B_{t+\tau,t+\tau+1}]$$

3. **The price of a long-term bond**

$$B_{t,t+n} = B_{t,t+1}B_{t,t+1,t+2}...B_{t+\tau,t+\tau+1}...B_{t+n-1,t+n}$$

Long-Bond Prices and Yields

$$B_{t,t+\tau,t+\tau+1} = E_t[e^{-r_{t+\tau}}]$$

$$B_{t,t+\tau,t+\tau+1} = e^{-\mu_{t+\tau}+\frac{var_t(r_{t+\tau})}{2}}$$

$$\mu_{t+\tau} = E_t[ln\, r_{t+\tau}]$$

Substitute in the equation for $B_{t,t+n}$ to obtain the long bond price.

Bond yields:

Define the yield by y_{t+n} in

$$B_{t,t+n} = e^{-y_{t+n}n}$$

and solve for

$$y_{t+n} = \frac{-ln(B_{t,t+n})}{n}$$

Long-Bond Prices and Yields

The price of a long-term bond

$$
\begin{aligned}
B_{t,t+3} &= B_{t,t+1} B_{t,t+1,t+2} B_{t,t+2,t+3} \\
&= e^{-r_t} E_t[e^{-r_{t+1}}] E_t[e^{-r_{t+2}}] \\
&= e^{-r_t} e^{-\mu_{t+1} + \frac{var_t(r_{t+1})}{2}} e^{-\mu_{t+2} + \frac{var_t(r_{t+2})}{2}}
\end{aligned}
$$

$$
\begin{aligned}
B_{t,t+3} &= e^{-[r_t + ka + (1-k) r_t + ka + (1-k) ka + (1-k)^2 r_t]} \\
&. \quad e^{\frac{1}{2}[2\sigma^2 + (1-k)^2 \sigma^2]} \\
&= e^{-[r_t [1 + (1-k) + (1-k)^2]} e^{-[2ka + (1-k) ka]} \\
&. \quad e^{\frac{1}{2}[2\sigma^2 + (1-k)^2 \sigma^2]}
\end{aligned}
$$

Long-Bond Prices and Yields

$$
\begin{aligned}
B_{t,t+n+1} &= e^{-[r_t[1+(1-k)+...+(1-k)^n]} \\
&. \quad e^{-[nka+(n-1)(1-k)ka+...+(1-k)^{n-1}ka]} \\
&. \quad e^{\frac{1}{2}[n\sigma^2+(n-1)(1-k)^2\sigma^2+...(1-k)^{2(n-1)}\sigma^2]}
\end{aligned}
$$

Using geometric progressions 3 and 4:

$$
\begin{aligned}
B_{t,t+n+1} &= e^{-r_t\left\{\left[\frac{1-(1-k)^{n+1}}{k}\right]\right\}} \\
&. \quad e^{-a\left\{n-(1-k)\left[\frac{1-(1-k)^n}{k}\right]\right\}} \\
&. \quad e^{\frac{\sigma^2}{2}\left\{\frac{n-(1-k)^2\left[\frac{1-(1-k)^{2n}}{1-(1-k)^2}\right]}{1-(1-k)^2}\right\}}
\end{aligned}
$$

Calibrating the Model to the Current Term Structure

Hull and White suggest the following generalisation:

$$r_{t+1} - r_t = k(a_t - r_t) + \varepsilon_{t+1}$$

This is equivalent to

$$r_{t+1} = r_t + \theta_t + k(a - r_t) + \varepsilon_{t+1}$$

where θ_t is a time dependent drift.

With this adjustment, the model can be fitted exactly to the current term structure of bond forward prices.

A Two-factor Extension

- **1-factor model does not reflect term-structure of volatility**
- **2-factor model has a stochastic central tendency**
- **Let**

$$
\begin{aligned}
r_{t+1} - r_t &= k_1(a_t - r_t) + y_t + \varepsilon_{t+1} \\
y_t - y_{t-1} &= -k_2 y_{t-1} + \nu_t
\end{aligned}
$$

k_1 **and** k_2 **are rates of mean reversion**

ε_{t+1} **and** ν_t **are a drawings from normal distributions** $(0, \sigma_1)$, $(0, \sigma_2)$

A Two-factor Extension

After successive substitution:

$$\begin{aligned}
r_{t+3} &= (1-k_1)^3 r_t + k_1[a_{t+2} + (1-k_1)a_{t+1} \\
&+ (1-k_1)^2 a_t] \\
&+ y_t[(1-k_1)^2 + (1-k_1)(1-k_2) + (1-k_2)^2] \\
&+ (1-k_1)^2 \varepsilon_{t+1} + (1-k_1)\varepsilon_{t+2} + \varepsilon_{t+3} \\
&+ (1-k_2)\nu_{t+1} + \nu_{t+2}
\end{aligned}$$

If $var(\varepsilon_{t+i}) = \sigma_1^2$ **and** $var(\nu_{t+i}) = \sigma_2^2$**,**

$$\begin{aligned}
var(r_{t+n}) &= \sigma_1^2 \left[\frac{1-(1-k_1)^{2n}}{1-(1-k_1)^2}\right] \\
&+ \sigma_2^2 \left[\frac{1-(1-k_2)^{2(n-1)}}{1-(1-k_2)^2}\right]
\end{aligned}$$

Annualised st.dev.

$$std_t(r_{t+n}) = \sqrt{var_t(r_{t+n})/n}$$

Gaussian Model: Advantages and Disadvantages

- Simple model
  - Relatively easy to program (using a binomial tree)
  - Can capture the current term structure
  - Two-factor model can reflect term structure of volatilities
- Yields analytical (zero-coupon) bond prices
  - Closed form expression for bond price
  - Bond prices are lognormal
  - Options on bonds priced with BS
- But, short rates may not normally distributed
- Two-factor model may not yield realistic swaption prices

Lecture 2: Lognormal Models

- **Types of Lognormal Models**
  - **One-Factor BK spot-rate model**
  - **Two-factor PSS spot-rate model**
  - **Forward rate models**
- **One-Factor BK model**
  - **Short rate follows a mean reverting, lognormal process**
  - **Under risk-neutral measure**
  - **Constructed using HSS (1995) method**
- **Two-Factor PSS model**
  - **Short rate follows a mean reverting, lognormal process**
  - **With stochastic central tendency**
  - **Constructed using PSS (2003) method**

Short-Rate Lognormal Model

Assume that:

$$ln\, r_{t+1} - ln\, r_t = k(a_t - ln\, r_t) + \varepsilon_{t+1}$$

short rate of interest, r_t

long-term mean of log of short rate, a_t

rate of mean reversion, k**,** $0 < k < 1$

ε_{t+1}**, drawing from a normal distribution,** $E_t(\varepsilon_{t+1}) = 0$**,** $var_t(\varepsilon_{t+1}) = \sigma^2$**.**

Hence,

$$ln\, r_{t+1} = ka_t + (1-k)ln\, r_t + \varepsilon_{t+1} \quad (1)$$

Equation (1) holds for all t

Mean and Variance of $ln\, r_{t+n}$ $(a_t = a)$

Mean of $ln\, r_{t+n}$:

$$E_t(ln\, r_{t+n}) = a[1-(1-k)^n] + (1-k)^n ln\, r_t$$

Variance of $ln\, r_{t+n}$:

$$var_t(ln\, r_{t+n}) = \sigma^2 \left[\frac{1-(1-k)^{2n}}{1-(1-k)^2}\right]$$

Annualised volatility

$$std_t(ln\, r_{t+n}) = \sqrt{var_t(ln\, r_{t+n})/n}$$

Mean of r_{t+n}:

$$E_t(r_{t+n}) = e^{E_t(ln\, r_{t+n}) + \frac{var_t(ln\, r_{t+n})}{2}}$$

Two-Factor BK Model

$$\begin{aligned} ln\, r_{t+1} - ln\, r_t &= k_1(a_t - ln\, r_t) + \ln\ y_t + \varepsilon_{t+1} \\ ln\, y_t - ln\, y_{t-1} &= -k_2 \ln\ y_{t-1} + \nu_t \end{aligned}$$

- y_t **is a 'premium' factor**
- **a shock to the futures rate**

Mean and Volatility of r_{t+n}

Since r_{t+n} is lognormal, the mean is given by

$$E_t(r_{t+n}) = e^{E_t(ln\, r_{t+n}) + \frac{var_t(ln\, r_{t+n})}{2}}$$

Example: 2-Factor BK model

$t = 0$

short rate, $r_0 = 0.05$

long term mean, $a = 0.06$

mean reversion, $k_1 = 0.15$

$\sigma_1 = 0.2$

mean reversion, $k_2 = 0.2$

$\sigma_2 = 0.15$

$\theta_t = 0$

$$E_0(r_5) = e^{-2.894 + \frac{0.168^2}{2}} = 0.0602$$

Volatility = 18.32%

Implementing the BK and 2-Factor BK Models

- **One-Factor BK spot-rate model**
  - **Fitting the mean of the process**
    - **Using futures Libor quotes**
    - **Iterative calibration to forward bond prices**
  - **Calibrating to cap volatilities**
    - **Generalise model using $\sigma(t)$**
  - **Recombining tree using HSS (1995)**
- **Two-factor BK model**
  - **PSS implementation**
    - **Recombining tree in two dimensions**
  - **G2++ model, Brigo and Mercurio**

HSS Implementing the BK Model

- **HSS (1995) build a binomial approximation to a lognormal process**
- **A re-combining tree with mean reversion and time-dependent voatility**
- **Conditional probabilities in the tree**
- **Method can be applied to interest rate process**

HSS Implemention of the BK Model

HSS Method

- **First build a process for x_t, where** $E_0(x_t) = 1$
- **Assume mean reversion and volatility same as in the interest-rate process**
- $$ln\, x_{t+1} = a_{x,t} + (1-k) ln\, x_t + \epsilon_{t+1}$$

$$\begin{aligned} a_{x,t} &= E_0[ln\, x_{t+1}] - (1-k)E_0[ln\, x_t] \\ &= \frac{-var_0[ln\, x_{t+1}]}{2} - (1-k)\frac{-var_0[ln\, x_t]}{2} \end{aligned}$$

- **The conditional probabilities, q_t depend on $a_{x,t}$ and k**
- **Scale the process using futures rates to obtain approximation to**

$$ln\, r_{t+1} = \theta_t + (1-k) ln\, r_t + \epsilon_{t+1}$$

HSS Implemention of the BK Model: an Example

- **Futures rates**

$$
\begin{aligned}
h_{0,1} &= 5.0\% \\
h_{0,2} &= 5.0\% \\
h_{0,3} &= 5.0\% \\
h_{0,4} &= 5.0\%
\end{aligned}
$$

- **Volatility (constant):** 10%
- **Mean reversion** $k = 0.2$
- **Cap Vols**

$$
\begin{aligned}
t &= 1 : 0.1 \\
t &= 2 : 0.0906 \\
t &= 3 : 0.0827 \\
t &= 4 : 0.0760
\end{aligned}
$$

Implemention of the 2-Factor BK Model
The PSS (2002) method

- **First build a process for x_t, y_t, where** $E_0(x_t) = 1,\ E_0(y_t) = 1$
- **Assume mean reversion and volatility same as in the interest-rate process**
- $$\begin{aligned} ln\, x_{t+1} &= a_{x,t} + (1 - k_1) ln\, x_t + ln\, y_t + \epsilon_{t+1} \\ ln\, y_{t+1} &= a_{y,t} + (1 - k_2) ln\, y_t + \nu_{t+1} \end{aligned}$$
- **The conditional probabilities, $q_{x,t}$ depend on $a_{x,t}$, k_1, and $ln\, y_t$**
- **Scale the process using futures rates to obtain approximation to**

$$ln\, r_{t+1} = \theta_t + (1 - k_1) ln\, r_t + \epsilon_{t+1}$$

Implemention of the 2-Factor BK Model
The G2++ method

Brigo and Mercurio suggest a transformation of the (r_t, y_t) process:

$$
\begin{aligned}
r_{t+1} - r_t &= \theta_t - k_1 r_t + y_t + \varepsilon_{t+1} \\
y_{t+1} - y_t &= -k_2 y_t + \nu_{t+1}
\end{aligned}
$$

Define

$$
\begin{aligned}
x_t &= r_t + \frac{y_t}{k_2 - k_1} \\
\psi_t &= \frac{y_t}{k_1 - k_2}
\end{aligned}
$$

Then:

$$
\begin{aligned}
x_t - x_{t-1} &= \theta_t - k_1 x_t + \eta_{t+1} \\
\psi_t - \psi_{t-1} &= -k_2 \psi_t + \frac{\nu_{t+1}}{k_1 - k_2},
\end{aligned}
$$

$$
\eta_{t+1} = \varepsilon_{t+1} + \frac{\nu_{t+1}}{k_1 - k_2}
$$

Implemention of the 2-Factor BK Model
The G2++ method

$$
\begin{aligned}
cov(\eta_{t+1}, \nu_{t+1}) &= \frac{1}{k_2 - k_1} var(\nu_{t+1}) \\
&= \frac{1}{k_2 - k_1} \sigma_2^2
\end{aligned}
$$

2-Factor BK Model Conclusions

- **BK model is too simple**
- **LMM is complex and lacking intuition**
- **2-factor BK model is a good compromise**
- **Captures term structure of cap volatilities**
- **Generate scenarios for risk management**
- **Valuation of Bermudan swaptions, exotics**
- **3-factor extension to capture swaption vols**