Asset Allocation Strategy Given Non-Market Wealth and Background Risks

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Abstract

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We examine the effects of non-portfolio risks on optimal portfolio choice. Examples of non-portfolio risks include uncertain labor income, uncertainty about the terminal value of fixed assets such as housing and uncertainty about future tax liabilities . While some of these risks are added to portfolio value and have been amply studied, others are multiplicative in nature and have received far less attention. The simultaneous effect of both additive and multiplicative risks has hitherto not received attention and can explain some paradoxical choice behavior. We rationalize such behavior and show how non-portfolio risks might lead to seemingly U-shaped relative risk aversion for a representative investor, as found empirically by Aït-Sahalia and Lo (2000) and Jackwerth (2000).

1 Introduction

It is well known that non-portfolio assets play a role in deciding upon an investment strategy. For example, consider the choice of a portfolio of stocks and bonds by an agent investing for a pension. In addition to her portfolio, the investor may have non-market wealth in the form of housing, uncertain bequests, or ownership of private businesses which also yields an uncertain income. Moreover, at the future retirement date, the portfolio value annuitized at an uncertain rate will determine the pension.

In this paper, we analyse the optimal portfolio choice of an investor who also has risky non-market wealth and whose portfolio value may be subject to a multiplicative conversion risk at the investment horizon. In particular, we extend the standard asset allocation problem by focusing on three factors that affect the optimal portfolio choice: (1) the expected value (mean) of non-market wealth, (2) the riskiness of the non-market wealth, and (3) a multiplicative conversion risk. To some extent, the effect of each of these has been analysed separately in the literature. However, we are not aware of any research that has hitherto examined their combined effects on portfolio choice, which can be surprising.

The fact that that non-market wealth can have significant effects on asset allocation is well recognized. A non-stochastic income stream from non-market wealth effectively substitutes for bond returns in the portfolio, as pointed out by Bodie, Merton and Samuelson (1992). For example, an investor with substantial housing assets can afford to choose a more aggressive asset allocation strategy for her market related assets (Cocco (2005)). A positive labor income, even if risky, may have a similar effect as shown by Koo (1998), Viceira (2001), Cocco (2005), Cocco, Gomez and Maenhout (2005) and Polkovnichenko (2007). However, Heaton and Lucas (2000) show that entrepreneurial risk is associated with less market risk taking. Similarly, Cocco, Gomez and Maenhout (2005) point out that the possibility of disastrous labor income shocks can induce a more conservative asset-allocation strategy.

All of the above models consider only non-negative non-market wealth. When the risky non-market wealth can be negative, the effects can be quite intriguing. For example, consider the realistic case in which non-market wealth is risky and has a positive mean. We show that if negative realizations of non-market wealth are possible, then an individual whose underlying preferences exhibit constant relative risk aversion (CRRA) behaves in financial markets as if her risk aversion was U-shaped, exhibiting declining relative risk aversion (DRRA) at low wealth levels followed by increasing relative risk aversion (IRRA) at higher wealth levels. Such an observation is not innocuous. Indeed in a very well crafted empirical study, Aït-Sahalia and Lo (2000) back out the risk aversion of a representative agent using market data and they show that observed relative risk aversion exhibits such U-shaped behavior. A similar result was found independently by Jackwerth (2000).

With regards to the riskiness of the non-market wealth, the literature on background risk is particularly relevant. An independent, additive, zero-mean background risk tends to make an agent more risk averse towards an independent market risk, leading to an increase in the bond proportion in the portfolio of marketable assets. This is easily seen by using the notion of the derived utility, as introduced by Kihlstrom, Romer and Williams (1981) and Nachman (1981). Kimball (1993) and Gollier and Pratt (1996) derive conditions under which derived risk aversion increases when an additive background risk is introduced, which in turn leads to a higher bond proportion. Eeckhoudt, Gollier and Schlesinger (1996) derive conditions under which an increase in background risk raises derived risk aversion.

The effect of a multiplicative conversion risk is less well researched. In general, this effect depends critically on the relative risk aversion function of the agent (Franke, Schlesinger and Stapleton (2006)). If utility is of the CRRA class, then it is well known that a non-hedgeable multiplicative risk will have no effect on portfolio selection. However if the agent exhibits declining or increasing relative risk aversion, there may be significant effects of multiplicative risks on asset allocation. For example, Campbell and Viceira (2001) and Brennan and Xia (2002) analyse the effect of a multiplicative inflation risk on asset allocation.

Combining both a risky non-market wealth and a multiplicative conversion risk can lead to surprising results. For instance, suppose that non-market wealth has a positive mean. Adding risk to the non-market wealth alone induces more risk-averse behavior of the investor; whereas adding multiplicative conversion risk alone induces less risk-averse behavior. However, adding both risks simultaneously does not yield some convex combination of these two separate effects. Rather, it may yield an even stronger increase in risk aversion than the non-market wealth risk alone. Thus, the interaction of both background risks needs to be considered in modeling portfolio decisions. The purpose of this paper is to explore these combined effects.

Of course, there are numerous other important influences on optimal asset allocation, many of which have been analysed in the literature. Authors usually focus on one particular factor. Whereas we assume that non-market wealth is a non-tradable risky asset, Yao and Zhang (2005) derive an optimal dynamic asset allocation strategy assuming that housing is a tradable asset. Brennan, Schwartz and Lagnado (1997) derive optimal policies assuming several risky tradable assets with mean reverting returns. Campbell, Cocco, Gomes, Maenhout and Viceira (2001) allow returns to mean revert and to have time-varying equity premiums. Brandt (1999) solves the problem taking into consideration variables that forecast time-varying investment opportunity sets. Balduzzi and Lynch (1999) analyse the case with predictability of asset returns and transaction costs as do Lynch and Balduzzi (2000). Lynch (2001) characterizes the hedging demands induced by return predictability. Xia (2001) discusses the effects of papameter uncertainty on dynamic asset allocation within a learning framework while Barberis (2000) concentrates on the long run implications of predictability. Brandt, Goyal, Santa-Clara and Stroud (2005) simulate portfolio choice using learning processes about return predictability. Chacko and Viceira (2005) analyse the impact of stochastic volatility in incomplete markets on portfolio choice.

In this paper, in line with much of the literature, we assume the investor has underlying preferences exhibiting CRRA. We derive the agent's optimal demand for state-contingent claims assuming a perfect market which is also complete regarding tradable claims. Given risky non-market wealth and the conversion risk, it is the agent's derived risk aversion towards market risk that determines her optimal investment policy. We first analyse the combined effects of risky non-market wealth and conversion risk on the derived relative risk aversion of the agent. To focus on the effects of background risks, we assume that the market pricing kernel exhibits constant elasticity.

In order to relate these findings to the literature on asset allocation, we illustrate the optimal policy in a multi-period setting in which the agent invests at each date in a single stock and a risk-free bond. Consistent with constant elasticity of the pricing kernel, the stock price is assumed to follow a geometric Brownian motion (GBM). Thus, in the absence of non-market wealth and other imperfections, the "Merton strategy" of holding a constant proportion of stocks would be optimal. Following any market movements, the investor would simply rebalance her portfolio back to the original proportions. Given the terminal date demand for claims in the presence of the background risks, we solve for the dynamic asset allocation rule which replicates this set of claims and compare it to the Merton strategy.

The outline of this paper is as follows. In the following section, we set up the problem of maximising the expected utility of terminal wealth and determine the optimality conditions for portfolio choice given risky non-market wealth and a multiplicative conversion risk. In section 3, we derive results showing the impact of a multiplicative risk, given the existence of positive or negative risky non-market wealth. In Section 4, we illustrate our results by considering a numerical simulation of investment behavior in a state-contingent claims framework. We then decompose this demand for contingent claims into dynamic stock/bond allocation decisions in section 5, prior to concluding the paper.

2 Conditions for Portfolio Optimisation Given Non-Market Wealth and Background Risks

Consider a risk-inverse investor who maximizes the expected utility of wealth over a fixed investment horizon. We consider a pure investment problem and ignore intermediate consumption. We assume that preferences are derived over the terminal value of wealth. For the sake of concreteness, we assume that these preferences exhibit constant relative risk aversion, with marginal utility of wealth w given by $u'(w) = w^{-\gamma}$, where $\gamma > 0$ denotes the degree of relative risk aversion.[1]

The terminal wealth of the investor is composed of portfolio wealth, $\tilde{x}$ and non-market wealth $\tilde{z}$. In addition, we assume that the market return, $\tilde{x}$, is subject to a unit-mean multiplicative risk $\tilde{y}$. Hence, terminal wealth is given by

$$\tilde{w} = \tilde{x}\tilde{y} + \tilde{z}, \;\; E(\tilde{y}) = 1. \tag{1}$$

We further assume that the random variables $(\tilde{x}, \tilde{z}, \tilde{y})$ are independent. Terminal wealth is composed of the endogenous terminal portfolio wealth $\tilde{x}$, subject to the exogenous risk $\tilde{y}$, and the exogenous wealth $\tilde{z}$. The endogenous portfolio wealth, $\tilde{x}$, depends on $\widetilde{R}$, the exogenous gross return on the market. The optimal investment policy can be described by a demand function $x(R)$ derived from a static one-period model.

Some examples illustrate this problem setup. (1) The investor may convert her terminal portfolio wealth at the uncertain annuity rate into a lifetime annuity. Then $\tilde{z}$ represents income from non-portfolio wealth including in-

[1] As is well known in this case, utility takes the form $u(w) = \ln w$ for $\gamma = 1$. Otherwise, $u(w) = (1-\gamma)^{-1} w^{1-\gamma}$.

come from another job after retirement. This income is reduced by medical cost, insurance premia etc. and other regularly paid cost. (2) The investor living in a small country invests abroad and converts the terminal portfolio wealth denominated in foreign currency into home currency at the uncertain exchange rate $\tilde{y}$. Then $\tilde{z}$ represents non-portfolio wealth like housing, bequests etc., reduced by personal debt, in home currency. (3) $\tilde{y}$ may be a purchasing power index reflecting uncertain inflation. Then $\tilde{z}$ is non- market wealth in real terms, for example, the present value of a pension which is indexed to the purchasing power index.

Letting $\phi(R)$ denote the pricing kernel, i.e. the price per unit of probability for a contingent claim with a payout of 1 in "state" R, the investor's objective can be written as

$$\max_{x(R)} E[u(x(\widetilde{R})\tilde{y} + \tilde{z})], \text{ s.t. } E[\phi(\widetilde{R})x(\widetilde{R})] = x_0. \tag{2}$$

Let

$$\nu(x) \equiv E[u(x\tilde{y} + \tilde{z})|x)]$$

denote the derived utility of tradable wealth, x.[2] Then the optimization

[2]See Kihlstrom, Romer and Williams (1981), Nachman (1982).

problem can be re-written as

$$\max_{x(R)} E[\nu(x(\widetilde{R}))], \text{ s.t. } E[\phi(\widetilde{R})x(\widetilde{R})] = x_0. \tag{3}$$

The first-order conditions for optimizing (3) are the budget constraint together with the conditions

$$\nu'(x(R)) = \lambda\phi(R), \ \forall R, \tag{4}$$

where λ is the Lagrange multiplier.[3] Differentiating (4) with respect to R and then using (4) to replace λ in the result, we obtain

$$x'(R) = \frac{\phi'(R)/\phi(R)}{\nu''(x(R))/\nu'(x(R))}. \tag{5}$$

Letting $A_{z,y}(x) \equiv -x\nu''(x)/\nu'(x)$ denote the Arrow-Pratt measure of relative risk aversion for the derived utility, it follows from (5) that the optimal contingent claim $x(R)$ satisfies

$$[A_{z,y}(x)]^{-1} = -\frac{x'(R)\phi(R)}{x(R)\phi'(R)} = -\frac{d\ln x(R)}{d\ln\phi(R)}. \tag{6}$$

Using (6), it follows that

$$\frac{d\ln x}{d\ln R} = \left[-\frac{d\ln x}{d\ln\phi}\right]\cdot\left[-\frac{d\ln\phi}{d\ln R}\right] = [A_{z,y}(x)]^{-1}\cdot\left[-\frac{d\ln\phi}{d\ln R}\right]. \tag{7}$$

[3]Note that for a CRRA investor her marginal utility covers the whole set of positive numbers. Therefore, given the usual integrability conditions, the optimal solution is interior (see Back and Dybvig (1993)).

Condition (7) always holds, i.e. regardless of the existence of non-market wealth and multiplicative risk. But the relative risk aversion $A_{z,y}(x)$ depends on those risks and so the optimal portfolio policy does also.

In the absence of non-market wealth ($z \equiv 0$) and multiplicative risks ($y \equiv 1$), $A_{z,y}(x) = \gamma$ due to our assumption of CRRA utility for total wealth. Given that we also assume a geometric Brownian motion for the evolution of $\widetilde{R}$, the second term (i.e. the elasticity of the pricing kernel) is also a positive constant.[4] Then, $\ln x(R)$ is an increasing and linear function of $\ln R$. That is, the investor has a log-linear demand function for contingent claims. Moreover, this linear function is flatter, *ceteris paribus*, for a higher level of relative risk aversion. This case is our benchmark case when we analyse the investor's portfolio policy in the presence of non-market wealth and multiplicative risks.

[4]As it is in the Black-Scholes economy, for example.

3 Portfolio Policy Given Non-Market Wealth and Multiplicative Risk

In this section, we examine the effect of a combination of non-market wealth and a multiplicative background risk. We look at the case in equation (1) where total wealth is given by

$$\tilde{w} = \tilde{x}\tilde{y} + \tilde{z}, \qquad (1)$$

with $E(\tilde{z}) \equiv \overline{z}$. We use a number of results, established in the literature, for special cases of (1). Let $A_z(x) \equiv A_{z,y}(x),\ given\ y \equiv 1$, i.e. the derived relative risk aversion in the absence of multiplicative risk. We summarize the results as follows.

1. Non-Stochastic Non-Market Wealth $\tilde{z} \equiv \overline{z}$, $\tilde{y} \equiv 1$

   Intuitively, if investors have positive, non-stochastic non-market wealth, their portfolio policy will be more aggressive than in the absence of this wealth. In fact, we have the following results which are similar to those found in Bodie, Merton and Samuelson (1992):

   - <u>Result 1a</u> Non-stochastic, positive non-market wealth and no multiplicative background risk [Bodie, et.al.]

Let $\tilde{z} = \overline{z} > 0$ *and* $\tilde{y} \equiv 1$. *Then an investor with CRRA preferences acts towards the market risk like an investor with HARA preferences*[5] *and derived relative risk aversion,* $A_z(x) < \gamma$. *Also,* $A_z(x)$ *is increasing in* x.

- Result 1b Non-stochastic, negative non-market wealth and no multiplicative background risk [Bodie, et.al.]

  Let $\tilde{z} = \overline{z} < 0$ *and* $\tilde{y} \equiv 1$. *Then an investor with CRRA preferences acts towards the market risk like an investor with HARA preferences and derived relative risk aversion,* $A_z(x) > \gamma$. *Also,* $A_z(x)$ *is decreasing in* x.

2. Stochastic Non-Market Wealth, $\overline{z} \leq 0$, $\tilde{y} \equiv 1$

   There is an extensive literature (see Gollier (2001)) studying the effect of a non-positive-mean, independent background risk on derived risk aversion. This has direct implications for portfolio policy. The following result follows from Kimball (1993):

   - Result 2 Stochastic, non-positive-mean, independent $\tilde{z}$, $\tilde{y} \equiv 1$ [Kimball]

[5] For HARA (hyperbolic absolute risk aversion) -utility $v'(x) = (x + b)^{-\gamma}$ for some constant b and relative risk aversion equal to $\gamma/(1 + b/x)$.

A CRRA investor with relative risk aversion γ, who is subject to an independent background risk, $\tilde{z}$, with $\overline{z} \leq 0$, acts towards the market risk like an investor with relative risk aversion $A_z(x) > \gamma$.

3. Multiplicative Background Risk

The effect of a unit-mean, independent multiplicative background risk has been studied in Franke, Schlesinger and Stapleton (2006). Although, for a CRRA investor, a multiplicative risk, $\tilde{y}$, has no effect on the optimal portfolio policy, we have the following relevant results:

- Result 3 Multiplicative background risk [Franke, et.al.]

  a) *Suppose an investor with declining, convex relative risk aversion and relative risk aversion greater than 1 is subject to a unit-mean multiplicative background risk, then the investor becomes more risk averse towards an independent market risk.*

  b) *Suppose an investor with increasing, concave relative risk aversion and relative risk aversion greater than 1 is subject to a unit-mean multiplicative background risk, then the investor becomes less risk averse towards an independent market risk.*

In this paper we assume throughout an investor with CRRA preferences. However, the above result is relevant since, given non-market wealth, the investor may act like someone with declining or increasing relative risk aversion and hence react to a multiplicative background risk by acting in a more (or less) risk averse manner, as we show below.

3.1 The Effect of Multiplicative Background Risk Given Negative Expected Non-Market Wealth

We begin our analysis of the joint effects of multiplicative background risk and non-market wealth by considering the case where the expected value of non-market wealth is negative. For example, the investor has some debt whose precise value will be known and repaid at the investment horizon. Although this may not be the most common or relevant case, it is the one where the analysis is straightforward and the results are easily understood. Result 1b above shows that a CRRA investor, with relative risk aversion γ, and negative non-stochastic non-market wealth acts like a HARA investor with relative risk aversion greater than γ, which is declining in x. If non-market wealth is stochastic and has negative mean, we would expect the derived relative risk aversion, $A_z(x)$ again to exceed γ and to be declining.

This is confirmed in the following, where z_{min} denotes the lowest outcome of $\tilde{z}$.[6]

Lemma 1 *If* $E(\tilde{z}) \equiv \overline{z} \leq 0$ *and* $\sigma_z > 0$, *then the derived relative risk aversion* $A_z(x) \to \infty$ *as* $x \to z_{min}$ *and* $A_z(x)$ *is declining, convex and approaches* γ *as* $x \to \infty$.

Proof: See Appendix

Lemma 1 together with Result 3a has the following direct implication for the portfolio demand of a CRRA investor. Let $\frac{\partial lnx}{\partial lnR}$, $[\frac{\partial lnx}{\partial lnR}|\tilde{z}]$, $[\frac{\partial lnx}{\partial lnR}|\tilde{z}, y]$ denote the slope in the absence of non- market wealth and multiplicative background risk, [in the presence of non- market wealth, given no multiplicative background risk], [in the presence of non-market wealth and multiplicative background risk].

Proposition 1 *A random non-market wealth with non-positive expected value induces the investor to choose a more conservative portfolio, as compared to the case without non- market wealth, and if the investor's level of CRRA:* $\gamma \geq 1$, *then a multiplicative risk induces the investor to choose an even more*

[6]More formally, z_{min} denotes the highest level of z such that $Prob(\tilde{z} \geq z_{min}) = 1$.

conservative portfolio, i.e.

$$\frac{d\ln x}{d\ln R}|\tilde{z},\tilde{y} < \frac{d\ln x}{d\ln R}|\tilde{z} < \frac{d\ln x}{d\ln R}, \quad \forall x.$$

Proof: See Appendix.

Proposition 1 follows from Lemma 1 and Result 3a. In Proposition 1, we define a 'more conservative' portfolio as one where the demand curve for contingent claims, $\frac{d\ln x}{d\ln R}$, has a flatter slope everywhere, i.e. the portfolio payoff is less sensitive to the market return. The right-hand inequality states that (non- positive) non- market wealth induces the investor to choose a more conservative portfolio. This follows from Lemma 1. The left-hand inequality states that the investor becomes even more conservative when subject, in addition, to a multiplicative background risk. This latter result stems from the fact that for the derived utility function given non-market wealth, $A_z(x) \geq \gamma \geq 1$ and $A_z(x)$ is declining and convex. The conclusion in the Proposition therefore follows directly from Result 3a above.

In this case, where $\overline{z} < 0$, all three effects of $\overline{z}$, σ_z, and $\tilde{y}$ work in the same direction. $\overline{z} < 0$ by itself induces a more conservative policy. This is re-enforced by the risk of non-market wealth, $\sigma_z > 0$. Also the existence of a multiplicative risk $\tilde{y}$ again reinforces this effect.

Assume a dynamically complete market for the market return. Then a given contingent claim demand can be obtained using only stocks and bonds. This is a practical way in which expected utility can be maximised over a long horizon. We therefore state, without proof, the following corollary to Proposition 1:

Corollary 1 *Assume the conditions of Proposition 1 and a dynamically complete market for the market return, but no tradability of the background risks. Then the investor reacts to the existence of non-market wealth (with non-positive mean) by reducing the proportion of market wealth invested in stocks (stock proportion) at each point in time and in each state. If the investor is also exposed to a multiplicative risk, then she further reduces her stock proportion, at each point in time and in each state.*

3.2 The Effect of Multiplicative Background Risk Given Positive Non-Market Wealth

For most investors, a more likely scenario is that the mean of non-market wealth is positive. We first consider the case where non-market wealth is always positive, but risky. The non-market wealth could come from a number of possible sources: labor income, real estate, or bequests, for example. How should you invest if you expect a large bequest sometime in the future, but have little idea as to the size of the bequest? The effect of the non-market wealth clearly depends on its risk. If the risk is small, in comparison with the expected value, then the derived risk aversion, $A_z(x) < \gamma$, and Result 1 above implies that $A_z(x)$ is increasing in x.

If the risk is large, but the minimum value of non-market wealth is positive, it is again likely that $A_z(x) < \gamma$ and increasing in x. However, this is not always the case.[7] The following Lemma characterizes $A_z(x)$ for $z_{min} > 0$.

Lemma 2 *Suppose non-market wealth is strictly positive, $z_{min} > 0$.*

1. *$A_z(x) \to 0$ as $x \to 0$ and $\exists(x^o, x^{oo})$ with $x^{oo} > x^o$ such that, for $x \leq x^o$ and for $x \geq x^{oo}$, $A_z(x)$ is increasing and concave. $A_z(x)$ approaches γ as $x \to \infty$. Also, $A_z(x) < \gamma$ for all x.*

[7] To prove that $A_z(x)$ may have a local maximum and a local minimum, it suffices to consider an example. Let $\tilde{z}$ be distributed symetrically around the mean of 30 according to (1, 6.25%; 20, 25%; 30, 37.5%; 40, 25%; 59, 6.25%). In each pair the first number denotes the realisation of z and the second number its probability. Then for $\gamma = 3$, $A_z(x)$ attains a local maximum at $x = 6$ and a local minimum at $x = 25$.

2. *If σ_z is sufficiently small, then $A_z(x)$ is increasing and concave for all x.*

Proof: See Appendix.

Lemma 2 part 2. establishes that the derived relative risk aversion, $A_z(x)$, is increasing and concave when the risk of non-market wealth is small. This mirrors the previous result in Lemma 1 for the case where non-market wealth is negative. Lemma 2 part 1. shows that the story is not so straightforward when σ_z is large. There are ranges of x over which $A_z(x)$ is increasing and concave, however it is possible that this is not true over some intermediate range.

The implications of Lemma 2 for portfolio policy are now stated in the following Proposition. Let y_{min} and y_{max} denote the lowest and highest possible outcomes of y. [8] Hence $x \leq x^o/y_{max}$ implies $xy \leq x^o, \forall y$ and $x \geq x^{oo}/y_{min}$ implies $xy \geq x^{oo}, \forall y$.

Proposition 2 *Suppose non-market wealth is strictly positive, $z_{min} > 0$.*

1. *Non-market wealth induces the investor to behave in a less conservative*

[8] More formally, y_{min} [y_{max}] denotes the highest [lowest] level of y such that prob($\tilde{y} \geq y_{min}$) = 1 [prob($\tilde{y} \leq y_{max}$) = 1]

manner

$\frac{d\ln x}{d\ln R}|\tilde{z} > \frac{d\ln x}{d\ln R}$, $\forall x$.

This effect is reinforced by the existence of a multiplicative risk for $x \leq x^o/y_{max}$ *and for* $x \geq x^{oo}/x_{min}$ *provided that* $A_z(xy) \geq 1, \forall xy$,

$$\frac{d\ln x}{d\ln R}|\tilde{z}, \tilde{y} > \frac{d\ln x}{d\ln R}|\tilde{z}, \; \forall \, x \leq x^o/y_{max}, x \geq x^{oo}/y_{min}.$$

2. *If* σ_z *is sufficiently small, then the last inequality holds for every* x, *given* $A_z(x,y) \geq 1 \; \forall xy$.

Proof:

The Proposition follows directly from Lemma 2 and Result 3b.

As expected, an investor with positive non-market wealth follows a less conservative portfolio policy, in the absence of multiplicative risks. This follows since positive non-market wealth acts as a substitute for bonds. When the investor is also subject to a multiplicative background risk, she tends to become even less conservative. Hence the portfolio choice effects of the two risks tend to reinforce each other.

Again, we simply state the following corollary, without proof:

Corollary 2 *Assume strictly positive non-market wealth and a dynamically* *complete market for the market return. Then the investor reacts to the existence of non-market wealth by increasing her stock proportion, at each point in time and in each state. If the investor is also exposed to a multiplicative risk, then she further increases her stock proportion, provided that non-market wealth risk is sufficiently small and relative risk aversion* $A_z(xy)$ *exceeds 1.*

Note that the second sentence of this corollary only applies in the case of Proposition 2.2 and not in the case of Proposition 2.1, where the effect on portfolio policy is more complex.

3.3 The Effect of Multiplicative Background Risk Given Positive Expected Non-Market Wealth, where the Minimum Non-Market Wealth is Negative

Perhaps the most likely case for many investors is where the expected value of non-market wealth is positive, but there is some chance that it might turn out to be negative. For example, the size of medical or education related liabilities may outweigh the positive benefits from bequests or property, in some future scenarios. Then the investor ends up being indebted at the investment horizon. Most investors have to consider the possibility of negative non-market wealth. In this case, the effect of non-market wealth on derived utility is more complex. The effect of a multiplicative background risk on derived relative risk aversion and on portfolio policy is correspondingly complex.

We begin the analysis as before by considering the possible effect of non-market wealth on derived relative risk aversion $A_z(x)$. We show:

Lemma 3 *Suppose* $E(\tilde{z}) > 0$ *and* $z_{min} < 0$. *Then* $A_z(x) \rightarrow \infty$ *as* $x \rightarrow -z_{min}$ *and* $\exists\,(x^o, x^{oo}, x^{ooo})$ *with* $x^o \leq x^{oo} \leq x^{ooo}$ *such that:*

- $A_z(x)$ *is declining and convex for* $x \leq x^o$,
- $A_z(x)$ *has a minimum at* x^{oo},
- $A_z(x)$ *is increasing and concave for* $x > x^{ooo}$.

Also, as $x \rightarrow \infty, A_z(x) \rightarrow \gamma$.

Proof: See Appendix.

The result in Lemma 3 shows that when the expected non-market wealth is positive, but the minimum non-market wealth negative, the behaviour of the derived relative risk aversion, $A_z(x)$, is range dependent. First, $A_z(x)$ is very high and declines to some level below γ. Finally, $A_z(x)$ approches γ as x becomes large. Hence there exists a minimum of $A_z(x)$. Hence it could be U-shaped, as illustrated by the numerical example in Figure 1. There is some empirical evidence in support of such a U-shape for relative risk aversion. In particular, Aït-Sahalia and Lo (2000) and Jackwerth (2000) use market data to show that the pricing kernel reflects a relative risk aversion function, of the representative investor, which is U-shaped.

In this numerical example, we consider a portfolio payoff x in the range (30,300). Non-market wealth is symmetrically distributed around its mean of 30 according to the distribution (-30, 6.25%; 0, 6.25%; 30, 37.5%; 60, 25%; 90, 6.25%). In each pair the first number denotes the realisation of z and the second number its probability. Hence $\tilde{z}$ has a standard deviation of 30 and a minimum realisation of -30. For $\gamma = 1.5$ the derived relative risk aversion $A_z(x)$ limits to infinity for $x = 30$ and to 1.5 for large x. In between there exists an x-range with $A_z(x) < \gamma$. $A_z(x)$ is declining and convex for $x < x^{oo} = 130$ where it reaches a minimum. After that it increases gradually to 1.5. For $x > x^{ooo} = 196$ it is increasing and concave.

The extent of the U-shape in the $A_z(x)$ function and its significance depend upon the values of $\overline{z} > 0$ and σ_z. If σ_z is small in comparison with the expected value, there may be only a small chance of $z < 0$ and consequently only a small x-range in which $A_z(x)$ declines. If σ_z is large in comparison with the expected value, the x-range in which $A_z(x)$ declines may be large.

Lemma 3 and Result 3 imply that the effect of a multiplicative risk on derived relative risk aversion is range dependent. The multiplicative risk raises derived relative risk aversion in the range $x \leq x^o/y_{max}$ and lowers it in the range $x \geq x^{ooo}/y_{min}$[9] Hence the U-shape of derived relative risk aversion is preserved under the multiplicative risk, as illustrated in Figure 1. The following proposition states the implications for portfolio choice.

Proposition 3 *Suppose $E(\tilde{z}) > 0$ and $z_{min} < 0$. Then non-market wealth induces the investor to behave in a more [less] conservative manner whenever $A_z(x) > [<] \gamma$.*

The addition of a multiplicative risk induces the investor to behave in a more [less] conservative manner for $x \leq x^o/y_{max}$ [$x \geq x^{ooo}/y_{min}$], provided that $A_z(xy) \geq 1, \forall xy$.

[9] This is also illustrated in Figure 1 assuming that y is binomial with realisations 0.7 and 1.3, each with probability 0.5. Then $xy_{min} + z_{min} = 0.7x - 30 > 0$ requires $x > 43$ so that $A_{z,y}(x)$ is defined only for $x > 43$.

Proposition 3 says first that in this case of positive expected non-market wealth with a negative minimum investors may act in either a more or less conservative manner given non-market wealth, since derived risk aversion $A_z(x)$ can be higher or lower than γ. Also, they may react to a multiplicative background risk by becoming either more or less conservative in their portfolio choice.

There are two interesting implications of Proposition 3. First, in the absence of multiplicative risk, we know that $\sigma_z > 0$ increases derived relative risk aversion and leads to a *more* conservative portfolio strategy. On the other hand, with $\sigma_z \to 0$ and $\overline{z} > 0$, Proposition 2 implies that a multiplicative risk, $\sigma_y > 0$ reduces derived relative risk aversion and makes the investor act in a *less* conservative manner. Since these two risks have opposite effects, one might expect the combination of the risks to lead to behaviour somewhere between these extremes. However, this need not be the case. The interaction of the two effects is somewhat complex. If σ_z gets large enough so that z_{min} becomes negative, the derived relative risk aversion, $A_z(x)$ changes from being increasing and concave (when $\sigma_z \to 0$) to being decreasing and convex, at least over some range of low values of x. If σ_z is large enough, the effect of a multiplicative risk will be to make the investor choose an even more conservative portfolio than in the absence of the multiplicative risk. Hence, we have an anomaly. The two risks in isolation act in opposite directions, but the two risks together act in the same direction, re-enforcing each other. This is the first counter-intuitive result. It illustrates the point that we cannot predict behaviour by considering the non-market wealth risk and the multiplicative risk separately. Whenever non-market wealth risk is large rendering z_{min} negative, it turns around the way in which the multiplicative risk affects derived relative risk aversion and portfolio policy.

The second interesting conclusion is a direct corollary of Proposition 3. We state it now as:

Corollary 3 *Assume the conditions of Proposition 3 and a dynamically complete market for the market return. Since, for large σ_z with $z_{min} < 0$, over some range of x the investor has declining, convex derived relative risk aversion and over another range increasing, concave derived relative risk aversion, then over the former range the investor will react to a multiplicative background risk by reducing her stock proportion and over the latter range by increasing her stock proportion.*

Proposition 3 and Corollary 3 show that we should expect complex reactions to the background risks, even if investors have CRRA utility. Observed behaviour can reflect increasing, declining, or U-shaped derived relative risk aversion.

4 Investment Simulations

In the following, we illustrate the previous results by some numerical examples. Our simulations are based around twelve different scenarios, detailed in Table 1. As we have seen above, the derived risk aversion of an agent and the resulting optimal portfolio demand depend upon three factors, the expected level of non-market wealth, the additive background risk associated with it, and the multiplicative background risk. Expected non-market wealth can be zero (the base case), positive (the most likely case) or negative. The additive background risk, measured by the standard deviation of non-market wealth, σ_z, can be positive or zero. The multiplicative background risk, measured by the standard deviation, σ_y, can also be positive or zero. Since all combinations are possible, there are a total of twelve possible cases.

In order to illustrate the effects of non-market wealth on portfolio choice, we now present numerical examples for each of the cases above. In Table 2 we summarize the data on which the numerical simulations are based. We assume that there are seven years with portfolio purchases made at dates $t = 0, 1, ..., 6$ and terminal wealth realized at date $t = 7$. The random values for $\widetilde{z}$ and $\widetilde{y}$ are also only realized at date $t = 7$. We thus have a static decision problem, with portfolio rules set at date $t = 0$. [10]

We approximate the geometric Brownian motion for the risky market return using a binomial approximation, where the mean excess return over any year is 5%. A risk-free one-year maturity bond pays 5% over each year. The annualized volatility of the market return is 20%.[11] From the binomial approximation, it follows that there are eight outcomes of the market return after seven years. If $r_{m,t}$ denotes the net market return in year t, then the gross market return over the seven years, R, is given by

$$R = \pi_{t=1}^{n}(1 + r_{m,t}).$$

When we derive the investor's static demand, $x(R)$, we present the demand

[10]Of course, a truly dynamic model with learning about $\widetilde{z}$ and $\widetilde{y}$ would be more realistic, but also more complex. Our point here is to show how these risks can affect observed portfolio-choice behavior, even in this simple setting.

[11]Hence, the elasticity of the pricing kernel, $\frac{d \ln \phi}{d \ln R}$ is equal to the equity premium, 0.05, divided by the squared market volatility, 0.04, or 1.25.

for each of the eight outcomes of R.

The investor has an initial wealth of 100, and has CRRA preferences with a coefficient of relative risk aversion, γ =1, 1.5, or 2. In the various cases shown below, the expected value of the non-market wealth, $\overline{z}$, takes on values of -20, 0 and 30.

The $\widetilde{z}$ risk in our simulation is calculated by $\widetilde{z}$ taking on a binomial distribution with the number of bifurcations, $n = 4$ and the probability of each up-move, $p = 0.5$, and where each up-move is modelled as an increase in $\widetilde{z}$ by 15, whereas a down-move is modelled as a decrease in $\widetilde{z}$ by 15. Hence, with $\tilde{z}$ risk, the lowest realisation of non-market wealth is negative, in all cases. The background risk $\widetilde{y}$ also is binomial with $n = 1$ and $p = 0.5$, with the initial value of $\widetilde{y}$ set at one, and with "success" indicating a 30 percent increase in $\widetilde{y}$, whereas "failure" indicates a 30 percent decrease in $\widetilde{y}$.

4.1 Fixed (Non-Stochastic) Non-Market Wealth

In Figure 2, we illustrate the optimal solution using the log-demand function from equation (1). We assume here that $\gamma = 1.5$. The optimal demand function is computed by solving the first-order condition for each state at date n, subject to the budget constraint embedded in (3). We take the three cases from Table 2, where non-market wealth is non-stochastic and multiplicative risk is absent, i.e. cases 1,5, and 9. In case 1, the expected non-market wealth is $\overline{z} = 0$. With $\overline{z} = 0$, the relative risk aversion $A(x) = \gamma = 1.5$ is a constant. Since the elasticity of the pricing kernel $(-\frac{d \ln \phi}{d \ln R})$ is also a constant, the optimal log-demand for state contingent claims is linear. This is an example of the Merton case.

Case 9 shows the effect of a positive expected non-market wealth. Here we assume that $\overline{z} = 30$. The resulting optimal demand function is steeper, reflecting the lower derived relative risk aversion. It is also concave reflecting the fact that, in this case, relative derived risk aversion is increasing. Case 5 shows the effect of a negative expected non-market wealth, where $\overline{z} = -20$. In this case, the demand curve is less steep and convex, reflecting greater risk aversion as well as decreasing relative risk aversion for the derived utility.

Using these cases as a starting point, we next examine the effect of either adding risk to non-market wealth, or adding a multiplicative background risk, or both.

4.2 Background Risks

In order to delineate the cases, we again consider three alternative scenarios in the presence of background risks: namely the cases where the expected non-market wealth is zero, negative or positive.

4.2.1 Zero expected non-market wealth

First consider the case where the non-market wealth is always zero, $\widetilde{z} \equiv 0$. As mentioned previously, adding the background risk $\widetilde{y}$ in this case has no effect on portfolio choice. This is seen in Figure 3 by noting that the base case (with $\widetilde{z} \equiv 0$) in case 1 yields identical results to case 2.

Now consider risky non-market wealth with zero expectation, $\overline{z} = 0$. This allows us to focus on pure risk effects. As is known in this case, the individual behaves in a more risk-averse manner when the non-market wealth is risky. This is seen in Figure 3 by comparing the base case (with $\widetilde{z} \equiv 0$) in case 1 with the case of a positive $\widetilde{z}$ risk in case 3. In this case, the demand for state contingent claims is flatter, due to the more-risk-averse portfolio choice. Moreover, the demand curve is no longer linear, but rather convex, due to the decreasing relative risk aversion in this case. This is as predicted by result 2 in section 3.

If we now consider both the $\widetilde{z}$ risk and the $\widetilde{y}$ risk, as existing concurrently, it follows from the appendix that behavior will be even more risk averse than under the $\widetilde{z}$ risk alone in the case where relative risk aversion is one or higher, $\gamma \geq 1$. Although the $\widetilde{y}$ risk in isolation does not affect behavior, it makes the investor worse off and here we see how it causes the investor to behave as if she were poorer and hence more sensitive to the additional $\widetilde{z}$ risk. Moreover, it follows in this case that $v(x) \equiv Eu(x\widetilde{y} + \widetilde{z})$ exhibits decreasing relative risk aversion.[12] This is illustrated in case 4 of Figure 3 and illustrates Proposition 1.

4.2.2 Negative expected non-market wealth

When $\overline{z} < 0$ and $\sigma_z = 0$, behavior appears to be more risk averse than CRRA would indicate as well as seeming to exhibit decreasing relative risk aversion. Adding the background risk $\widetilde{y}$ makes behavior seem even more risk averse. This shows up only slightly in Figure 4 in cases 5 and 6. This

[12] This follows by noting that $\widehat{v}(x) \equiv Eu(x + \widetilde{z})$ satisfies decreasing relative risk aversion, as shown in the appendix of this paper. Hence, from Franke, et al (2006, appendix), it follows that adding the $\widetilde{y}$ risk to obtain $v(x) \equiv Eu(x\widetilde{y} + \widetilde{z})$ preserves this property.

illustrates Results 1b and 3 above.

It also follows in Case 7 that adding the $\widetilde{z}$ risk, but in the absence of the $\widetilde{y}$ risk, makes behavior seem more risk averse and also seem to exhibit decreasing relative aversion, as is seen by comparing cases 5 and 7 in Figure 4. This illustrates Result 2 above.

If we include both the $\widetilde{z}$ risk and the $\widetilde{y}$ risk simultaneously, we see that the effects of more risk averse behavior and of decreasing relative risk aversion are magnified. Case 8 in Figure 4 illustrates this situation.

4.2.3 Positive expected non-market wealth

This case is perhaps the most interesting. When $\overline{z} > 0$, but background risks are absent, the individual appears to behave in a less risk-averse manner than CRRA would indicate. In addition, behavior appears to exhibit increasing relative risk aversion. This is the base case 9 in Figure 5, where the log-demand curve for contingent clams is slightly concave. If we add the multiplicative background risk $\widetilde{y}$, behaviour here becomes slightly less risk averse, as indicated by the slightly steeper demand curve. This is illustrated in case 10 and is as predicted by Result 3b.[13]

If a risk $\widetilde{z}$ is added to the non-market wealth in the absence of multiplicative $\widetilde{y}$ risk, the addition of $\widetilde{z}$ increases the level of risk aversion (case 11). Since non-market wealth is negative with positive probability, Lemma 1 applies. Hence, for low levels of marketable wealth relative risk aversion is higher than γ, then it declines to a level below γ and gradually approaches γ for high levels of marketable wealth. Hard to see in the diagram for case 11, for low values of $\ln R$ the log-demand curve for contingent claims is convex, indicating decreasing relative risk aversion. But for high levels of $\ln R$, the demand curve is slightly concave, indicating that relative risk aversion is increasing.

In case 12, we show the effects of including both the $\widetilde{z}$ risk and the $\widetilde{y}$ risk simultaneously. Although the effect of the $\widetilde{y}$ risk in isolation is to cause a decrease in observed risk aversion, the $\widetilde{y}$ risk also makes the individual more sensitive to the $\widetilde{z}$ risk. From Proposition 3 we know that for low levels of marketable wealth both risks reinforce each other making behavior appear more risk averse than in the presence of the $\widetilde{z}$ risk only. This is

[13]The property of increasing relative risk aversion need not always hold in this setting, but it does in this example.

evident in Figure 5 from the small slope of the case 12-curve in the range of low market returns. For high levels of marketable wealth, the $\widetilde{y}$ risk makes behavior appear less risk averse than in the presence of the $\widetilde{z}$ risk only. Hence in Figure 5 the slope of the case 12-curve is smaller than that of the case 10-curve in the range of high market returns.

5 Stock Proportions and Dynamic Allocation

In this section, we present our simulation results in a different form which better illustrates the impact of non-market wealth and multiplicative background risk on portfolio policy. Since the market return follows a binomial process and a riskless one-period bond exists, the market is dynamically complete. Therefore, it follows that any state-contingent claim demand $x(R)$ can be replicated with a period-by-period stock/bond asset allocation strategy. In the case where $A_{z,y}(x)$ is a constant, we know from Merton (1971) that the replicating strategy is to hold a constant proportion of wealth in stocks, throughout the period from time 0 to time n. However, in the general case, with derived utility $v(x) = Eu(x\widetilde{y} + \widetilde{z})$, the measure of relative risk aversion for v, $A_{z,y}(x)$, is not constant. Hence, the dynamic strategy is more complex.

At time n, the market return R and the state-contingent claim $x(R)$ have $n+1$ outcomes, indexed by $i = 0, 1, ..., n$. Moving back to time $n-1$, the market return has $n-1$ outcomes. Given a state i at time $n-1$, the market return can only move to one of two states at time n. It follows that there is a unique optimal stock proportion for each state at time $n-1$. The optimal dynamic strategy can be found by moving back through the binomial tree and solving for the stock proportions at each point of time and in each state. Let x_0 denote the initial wealth invested in the stock-bond portfolio and $x(R_i)$ the portfolio wealth at time n, given the gross market return R_i, $i = 0, 1, ..., n$. If $r_{m,t}$ is the market return in period t, and the risk-free rate is r_f, then

$$\frac{x(R_i)}{x_0} = \Pi_{t=1}^{n}[1 + \alpha_{i,t-1} r_{m,t} + (1 - \alpha_{i,t-1}) r_f], \ i = 0, 1, ..., n$$

is solved for the stock proportions, $\alpha_{i,t}$. Thus we obtain one unique optimal stock proportion for year 0, two unique stock proportions for year 1 and so on.

Table 3 shows the optimal stock proportions in year 0 and year 6, across different states, for the twelve different cases illustrated previously in Figures 2-5. The year-6 states are indexed by the number of down-ticks of the binomial process of the market return. Hence, state 0 is the highest market state and state 6 is the lowest. From condition (7), the proportion of stock reflects the degree of relative risk aversion. Hence, if the stock proportion is constant (declining) (increasing) across states, this indicates constant (declining) (increasing) derived relative risk aversion for market wealth. The results are shown for all the twelve different cases, which allow us to analyze the effects of the expected value of non-market wealth and its risk and the the multiplicative $\widetilde{y}$ risk, both separately and jointly.

Cases 1-4 show the effect of the two risks in the case of a zero-mean non-market wealth, $z = 0$. In the absence of both the non-market wealth risk and the multiplicative background risk, the investor follows the Merton strategy, investing 78% of her wealth in stocks in year 0 and also 78% in each state at year 6. When multiplicative background risk $\widetilde{y}$ risk alone is introduced (case 2), there is no effect on stock proportions. This is due to the fact that the utility is CRRA and $\tilde{z} \equiv 0$. The introduction of additive non-market wealth risk (case 3), reduces stock proportions and causes the proportions in year 6 to be state dependent, reflecting the convexity of the log-return function. The investor behaves towards the market risk like someone with declining relative risk averse (DRRA) utility. This in turn implies that the further introduction of the background $\widetilde{y}$ risk makes the investor choose even less stocks (case 4), as noted in Corollary 1.

Cases 5-8 show the effect of the two background risks individually and jointly in the case where the expected non-market wealth is negative, $\overline{z} < 0$. In all four cases investor behavior is consistent with DRRA, as stated in Lemma 1. This is illustrated by the stock proportions in year 6, which are higher in the high market states. Also, the effect of the non-market wealth and $\widetilde{y}$ risks is straightforward in this set of cases. Non-market wealth risk $\widetilde{z}$ alone reduces stock proportions, as does $\widetilde{y}$ risk. Also, the joint effect of the two risks together (case 8) is to reduce stock proportions even more.

The more complex and perhaps more relevant scenarios are illustrated in cases 9-12, where the expected value of non-market wealth is positive, $\overline{z} > 0$, but the lowest possible non-market wealth is negative. Here, in case 9, where both risks are zero, observed behavior exhibits increasing relative risk aversion, IRRA. In case 10, where the multiplicative background risk $\widetilde{y}$ is introduced, the effect is to increase the stock proportion, both in year 0 and in year 6. In this example, IRRA behavior is preserved under the $\widetilde{y}$ risk. However, the effect of introducing the non-market wealth risk $\widetilde{z}$ alone, in case 11, is to reduce the stock proportion and, given the chosen parameter values, to produce the U-shaped behavior for relative risk aversion as mentioned in the previous section. This is because the effect of the non-market wealth risk and the consequent precautionary premium outweigh the effect of the positive expected non-market wealth on the utility function in the low states.

This also explains why, in case 12, the compounding effect of the multiplicative $\widetilde{y}$ risk now reduces the stock proportion even further in the low states. However, as stated in Proposition 3, in the high states the multiplicative risk now increases the investment in stocks. This is because in the high states the precautionary premium for the $\widetilde{z}$ risk is very small. It is important to note however that the effects on derived relative risk aversion exhibited here depend upon the positive probability of a negative non-market wealth. If non-market wealth is always positive, then the effects on derived relative risk aversion are quite different.

The examples shown in Table 4 emphasize this point using sensitivity analysis on the parameter values. Cases 11a and 11b and 12a and 12b show the effect of varying the size of the additive risk, σ_z. Note that the lower risk in 11a induces IRRA derived utility. The higher $\tilde{z}$ risk in 11b induces DRRA derived utility. These cases are in contrast to case 11, where derived relative risk aversion is U-shaped. Of course, we would still observe U-shaped behavior if we expand the number of periods and, hence, the range of market returns. The corresponding response to the multiplicative risk is shown in case 12a, where stock proportions are higher than in the absence of the multiplicative risk (case 11a) in all states except for the worst state and in case 12b, where all stock proportions are lower than in the absence of the multiplicative risk (case 11b).

A further sensitivity analysis is carried out with respect to the coefficient of relative risk aversion (γ). 11c and 12c show that the effects are preserved, but dampened in the case of higher risk aversion. 11d and 12d show that the effects are preserved, but enhanced in the case of lower risk aversion.

In the figures 6-9, we illustrate graphically the optimal asset allocation strategy for a few of these cases over a five-year time interval. In Figure 6, we assume that expected non-market wealth is $\overline{z} = 30$, non-market wealth risk $\sigma_z = 0$ and the multiplicative risk $\sigma_y = 0.3$. This is case 10, in Table 3. The effect of the positive-mean non-market wealth, relative to the benchmark case, is to produce less risk averse behavior that exhibits IRRA and this is reflected in the dynamic asset allocation strategy shown. The investor starts with 96% invested in stocks at year 0. Then at year 1, this falls to 92% if the market moves up (0 down moves) and increases to 99% if the market moves down (1 down move). At year 2, the investor puts either

90%, 95% or 103% in stocks depending on the market state. Since there is an inverse relationship between the number of down-moves and the level of the market, the strategy reflects IRRA utility (a higher stock proportion as the market declines).

In Figure 7, we assume that expected non-market wealth is zero, $\overline{z} = 0$, the non-market wealth is risky, $\sigma_z = 30$, and the multiplicative background risk is $\sigma_y = 0.3$. This is case 4, in Table 3. The effect is to produce DRRA behavior and this is reflected in the dynamic asset allocation strategy shown. The investor starts with 55% invested in stocks at year 0. Then at year 1, this increases to 61% if the market moves up (0 down moves) and declines to 50% if the market moves down (1 down move). At year 2, the investor puts either 65%, 55% or 44% in stocks depending on the market state. Since there is an inverse relationship between the number of down-moves and the level of the market, the strategy reflects DRRA utility (a lower stock proportion as the market declines).

In Figures 8 and 9, we illustrate the optimal dynamic strategy for cases 11 and 12 in Table 3. Here, the positive $\overline{z}$ is balanced by a positive non-market wealth risk. In Figure 8 there is no multiplicative risk. Figure 8 reveals an interesting pattern of stock proportions. The $0d$ curve shows that as the market rises, the investor first invests more in stocks (DRRA) but then reduces it in later years (IRRA). Similar patterns are reflected in Figure 9, where $\sigma_y = 0.3$.

The resulting outcome shows an inverted U-shape allocation strategy. As the market starts to fall, the investor raises the stock proportion at first. But if the market continues falling, she starts to lower the stock proportion. This phenomenon follows by examining relative risk aversion at the appropriate wealth levels. Relative risk aversion is U-shaped here, and we are initially on the upward sloping part of the "U". If the market rises, observed behavior seems to exhibit increasing relative risk aversion (a lower percent in stock as wealth increases). However for downward movement in the market, risk aversion initially falls but then rises, as we pass the trough on the U-shaped relative risk aversion.

6 Conclusions

Portfolio selection is complicated by personal circumstances which can radically affect the asset allocation strategy of the investor. Here, we have analyzed the optimal strategy of a CRRA investor in a market where a single risky asset follows a geometric Brownian motion. The investor has stochastic non-market wealth and is also subject to a multiplicative background risk. If we only observe the portfolio choice of the investor, it might be difficult to observe anything that looks similar to her underlying CRRA preferences. The existence of non-market wealth may cause the investor to act as if her utility had increasing or declining relative risk aversion, depending on the size and risk of the non-market wealth. The response to a multiplicative background risk crucially depends upon the nature of any non-market wealth and its riskiness .

Consideration of the additive non-market wealth risk and the multiplicative background risk together in one model is important, since the combined effect can be quite different from the effect of one of these risks alone. Consider the case in which non-market wealth has positive expectation. Then the effect of the multiplicative risk alone is to increase investment in stocks, whereas the effect of this same risk may be to reduce investment in stocks when non-market wealth risk already exists . Ignoring the interaction effects between the risks can lead to incorrect predictions of the effect of background risks on portfolio policies.

In our model, resolution of the uncertainty surrounding the non-market wealth risk and multiplicative background risks only takes place at the horizon date. We solve what is essentially a single-period model for the optimal demand function for state-contingent claims. However, since the market for the risky asset is dynamically complete, this function can be represented by a dynamic asset-allocation strategy involving stocks and bonds. We find that this strategy is both time and state dependent. It follows that simple prescriptions for asset-allocation such as "lifestyle", which suggests a shift of assets from stocks to bonds as retirement approaches, is unlikely to be optimal.

Appendix

Proof of Lemma 1:

Let $A(x+z) \equiv -u''(x+z)x/u'(x+z)$. Then

$$\begin{aligned} A_z(x) &= \frac{E[-u''(x+z)]}{E[u'(x+z)]}x = E\left[\frac{u'(x+\tilde{z})}{Eu'(x+\tilde{z})}A(x+\tilde{z})\right] \\ &\equiv E^Q[A(x+\tilde{z})] = \gamma E^Q\left(\frac{x}{x+\tilde{z}}\right). \end{aligned} \tag{8}$$

$A_z(x) \to \infty$ for $x \to -z_{min}$, since $A(x+z_{min})$ goes to infinity. If $x \to \infty, A_z(x) \to \gamma$.

Claim 1: Given the conditions of Lemma 1, $A_z(x)$ is declining.

Proof of claim 1

Let $\overline{z} \equiv E(\tilde{z})$ and define $\widehat{v}(x) \equiv u(x+\overline{z})$. Then $\widehat{v}(x)$ exhibits standard risk aversion and either constant or decreasing relative risk aversion. Define $v(x) = Eu(x+\tilde{z})$. Then

$$v'(x) = E\widehat{v}'(x+\tilde{z}-\overline{z}) \equiv \widehat{v}'(x-\psi), \tag{9}$$

where ψ denotes Kimball's (1990) precautionary premium for $\tilde{z}-\overline{z}$.

Relative risk aversion for $v(x)$ is then calculated as

$$A_z(x) = -\frac{x\widehat{v}''(x-\psi)(1-\psi')}{\widehat{v}'(x-\psi)} = \widehat{A}(x-\psi)\cdot\frac{(1-\psi')}{1-\frac{\psi}{x}} \tag{10}$$

where

$$\widehat{A}(x-\psi) \equiv -\frac{\widehat{v}''(x-\psi)}{\widehat{v}'(x-\psi)}(x-\psi).$$

Since $\widehat{v}$ exhibits standard risk aversion, we know from Kimball (1993) that $\psi'(x)$ is negative. Moreover, from (9) we see that $\psi < x$, so that $1-\frac{\psi}{x} > 0$.

Differentiating (10)

$$A_z'(x) = \widehat{A}'(x-\psi) \cdot \frac{(1-\psi')^2}{1-\frac{\psi}{x}}$$
$$+\widehat{A}(x-\psi) \cdot \frac{(-\psi'')(1-\frac{\psi}{x}) + (1-\psi')(\frac{x\psi'-\psi}{x^2})}{(1-\frac{\psi}{x})^2}. \quad (11)$$

The first term on the right-hand side in (11) is non-positive by the assumptions. The second term is negative since $\psi'' > 0$, which follows from Franke et al (1998, Lemma 2). Hence, $A_z'(x) < 0$.

Claim 2: Given the conditions of Lemma 1, $A_z(x)$ is convex.

Proof of claim 2

First, we show that if u is a HARA-utility function with $\gamma > 0$, then $\varphi'(x) < 0$, $\varphi''(x) > 0$ and $\varphi'''(x) < 0$.

Franke et al (1998) have shown $\varphi'(x) < 0$ and $\varphi''(x) > 0$. Therefore $\varphi'''(x) < 0$ remains to be shown. For notational convenience, let $\nu = x + \overline{z}$ and

$\tilde{z} - \overline{z} = \sigma\tilde{\eta}$, where $\tilde{\eta}$ is a random variable with mean zero and unit variance.

We have

$$(\nu - \psi)^{-\gamma} = E[(\nu + \tilde{z} - \overline{z})^{-\gamma}]$$

or

$$\left(1 - \frac{\psi}{\nu}\right)^{-\gamma} = E\left[\left(1 + \frac{\sigma\tilde{\eta}}{\nu}\right)^{-\gamma}\right].$$

For a given η-distribution, it follows that

$$\frac{\psi}{\nu} = f\left(\frac{\sigma}{\nu}\right)$$

or

$$\psi = \nu f\left(\frac{\sigma}{\nu}\right).$$

Differentiating w.r.t. ν

$$\begin{aligned}\psi_\nu &= f\left(\frac{\sigma}{\nu}\right) + \nu f'\left(\frac{\sigma}{\nu}\right)\frac{-\sigma}{\nu^2} \\ &= f\left(\frac{\sigma}{\nu}\right) - f'\left(\frac{\sigma}{\nu}\right)\frac{\sigma}{\nu}\end{aligned}$$

and differentiating w.r.t. σ

$$\psi_\sigma = \nu f'\left(\frac{\sigma}{\nu}\right)\frac{1}{\nu} = f'\left(\frac{\sigma}{\nu}\right)$$

Hence

$$\psi_\nu = f\left(\frac{\sigma}{\nu}\right) - \psi_\sigma\frac{\sigma}{\nu}.$$

Differentiating again w.r.t. ν

$$\psi_{\nu\nu} = \frac{\partial \psi_\nu}{\partial \frac{\sigma}{\nu}} \frac{\partial \frac{\sigma}{\nu}}{\partial \nu} = \frac{\partial \psi_\nu}{\partial \frac{\sigma}{\nu}} \left(-\frac{\sigma}{\nu^2}\right)$$

and differentiating again w.r.t. σ

$$\psi_{\nu\sigma} = \frac{\partial \psi_\nu}{\partial \frac{\sigma}{\nu}} \frac{\partial \frac{\sigma}{\nu}}{\partial \sigma} = \frac{\partial \psi_\nu}{\partial \frac{\sigma}{\nu}} \left(\frac{1}{\nu}\right).$$

It follows that

$$\psi_{\nu\nu} = \psi_{\nu\sigma} \left(-\frac{\sigma}{\nu}\right)$$

which is a function of σ/ν. Hence,

$$\psi_{\nu\nu\nu} = \frac{\partial \psi_{\nu\nu}}{\partial \frac{\sigma}{\nu}} \frac{\partial \frac{\sigma}{\nu}}{\partial \nu} = \frac{\partial \psi_{\nu\nu}}{\partial \frac{\sigma}{\nu}} \frac{-\sigma}{\nu^2}$$

and

$$\psi_{\nu\nu\sigma} = \frac{\partial \psi_{\nu\nu}}{\partial \frac{\sigma}{\nu}} \frac{\partial \frac{\sigma}{\nu}}{\partial \sigma} = \frac{\partial \psi_{\nu\nu}}{\partial \frac{\sigma}{\nu}} \frac{1}{\nu} > 0.$$

Positivity of $\psi_{\nu\nu\sigma}$ is shown in Franke et al (1998, Lemma 3), and hence

$$\psi_{\nu\nu\nu} = \psi_{\nu\nu\sigma} \left(-\frac{\sigma}{\nu}\right) < 0.$$

This is equivalent to $\psi''' < 0$, q.e.d.

Now we show that $A_z''(x) > 0$.

In the HARA case,

$$A_z(x) = \gamma \frac{1 - \psi'(x)}{x + \overline{z} - \psi(x)} x$$

Differentiating w.r.t. x we have

$$A_z'(x) = -\gamma \frac{\psi''(x)x}{x + \overline{z} - \psi(x)} + \frac{A_z(x)}{\gamma x}[\gamma - A_z(x)]. \tag{12}$$

Differentiating the first term of (12) w.r.t. x yields

$$-\gamma \frac{\psi'''(x)x}{x + \overline{z} - \psi(x)} + \frac{\psi''(x)}{x + \overline{z} - \psi(x)}[A_z(x) - \gamma], \tag{13}$$

which is positive, since $\psi'''(x) < 0$, as shown above. Also, the second term in (12) clearly increases with x, since $A_z'(x) < 0$. Hence, $A_z''(x) > 0$.□

Proof of Lemma 2

For a small σ_z and positive $\overline{z}$, relative risk aversion is similar to that of a HARA function without NMW risk. It follows that the relative risk aversion of the derived utility function is increasing and concave. This proves Lemma 2.1.

To prove Lemma 2.2, first, from equation (8) $A_z(x) < \gamma, \forall x$. Second, we show that $A_z(x) \to 0$ for $x \to 0$. As

$$A_z(x) = xE^Q \left(\frac{\gamma}{x + \tilde{z}} \right)$$

and $x + \tilde{z} > 0$, $A_z(x) \to 0$ for $x \to 0$.

Next, we show that $A_z(x)$ is increasing and concave for $x \leq x^o$. In equation (12), the first term goes to zero for $x \to 0$, while the second term is positive. The latter follows from

$$\frac{A_z(x)}{\gamma x} = E^Q \left(\frac{1}{x + \tilde{z}} \right) > 0$$

and $[\gamma - A_z(x)] \to \gamma$ for $x \to 0$. Hence, $A_z'(x) > 0$ for $x \to 0$.

$A_z''(x)$ is given by (13) plus the derivative of $E^Q(x + \tilde{z})^{-1}[\gamma - A_z(x)]$ with respect to x. For $x \to 0$, the first term in (13) goes to zero while the second term is negative. Also, $E^Q(x+\tilde{z})^{-1}[\gamma - A_z(x)]$ declines in x since each factor is positive and declining. Hence, $A_z''(x) < 0$ for $x \leq x^o$.

Third, we show that $A_z(x)$ is increasing and convex for $x \geq x^{oo}$. Given a CRRA-utility function,

$$\begin{aligned} A_z(x) &= \gamma \, \frac{E(x + \tilde{z})^{-\gamma-1}}{E(x + \tilde{z})^{-\gamma}} \, x \\ &= \gamma \, \frac{[x + \overline{z} - \varphi]^{-\gamma-1}}{[x + \overline{z} - \varphi]^{-\gamma}} \left(1 - \frac{\partial \varphi}{\partial x}\right) \, x \end{aligned}$$

Since we now consider large values of x, we may regard $\tilde{\varepsilon} = \tilde{z} - \overline{z}$ as a small risk in the sense of Pratt (1964). Technically, divide $x + \tilde{z}$ by a large positive constant c so that $\sigma(\tilde{\varepsilon}/c)$ is a small risk. Then, dropping c for notational simplicity, for a large x the precautionary premium is given by

$$\varphi(\tilde{\varepsilon}|x) = \frac{\gamma+1}{x+\overline{z}}\,\frac{\sigma^2(\tilde{\varepsilon})}{2}$$

so that

$$\frac{\partial\varphi}{\partial x} = -\frac{\varphi}{x+\overline{z}}$$

Hence,

$$\begin{aligned} A_z(x) &= \gamma\,\frac{1+\varphi/(x+\overline{z})}{x+\overline{z}-\varphi}\,x \\ &= \gamma\,\frac{x}{x+\overline{z}}\,\frac{x+\overline{z}+\varphi}{x+\overline{z}-\varphi} \\ &= \gamma\,\frac{x}{x+\overline{z}}\,\frac{1+\frac{\gamma+1}{2}\sigma^2(\tilde{\varepsilon})(x+\overline{z})^{-2}}{1-\frac{\gamma+1}{2}\sigma^2(\tilde{\varepsilon})(x+\overline{z})^{-2}} \end{aligned}$$

For large values of x, the second fraction converges much faster to 1 than the first fraction because the second fraction depends on $(x+\overline{z})^{-2}$. Therefore, $A_z(x) \to \gamma\frac{x}{x+\overline{z}} < \gamma$ and finally to γ. Hence, $A_z(x)$ is increasing for large values of x.

Finally, in the last equation the first fraction is concave in x while the second is convex. Again, for high values of x, the first fraction "dominates" the second, which moves much faster to 1. Hence, $A_z(x)$ is increasing and concave $x \geq x^{oo}$. □

Proof of Lemma 3

$$A_z(x) \quad = \quad E\left[\frac{u^{'}(x+\tilde{z})}{Eu^{'}(x+\tilde{z})}\;\frac{x}{x+\tilde{z}}\right] \equiv \gamma E^Q\left(\frac{x}{x+\tilde{z}}\right).$$

Hence, $A_z(x) \to \infty$ for $x \to \underline{x} = -z_{min}$ where z_{min} is the minimal value of z with positive probability (density). Also, $A_z^{'}(x) \to -\infty$ for $x \to \underline{x}$. Also, $A_z(x) \to \gamma$ for $x \to \infty$. Since $A_z(x) > 0$ for $x > \underline{x}$, $A_{z,\varepsilon}^{''}(x) > 0$ is implied for some range $x \in (\underline{x}, x^{\circ})$ with $x^{\circ} > \underline{x}$.

Next, by the same argument as used in Lemma 2.2, $A_z(x)$ is increasing and concave for large values of x. This implies a) that there exists a finite $x^{\circ\circ}$ at which $A_z(x)$ attains a minimum, and b) there exists some $x^{\circ\circ\circ} > x^{\circ\circ}$ so that $A_z(x)$ is increasing and concave in x for $x > x^{\circ\circ\circ}$.

Table 1: Twelve Cases of Expected Non-Market Wealth, Additive Background Risk and Multiplicative Background Risk

	Non-stochastic non-market wealth No multiplicative risk	Non-stochastic non-market wealth Multiplicative risk	Stochastic non-market wealth No multiplicative risk	Stochastic non-market wealth Multiplicative risk
Zero-mean non-market wealth	Case 1: $\overline{z} = 0$, $\sigma_y = 0$, $\sigma_z = 0$	Case 2: $\overline{z} = 0$ $\sigma_y > 0$, $\sigma_z = 0$	Case 3: $\overline{z} = 0$ $\sigma_y = 0$, $\sigma_z > 0$	Case 4: $\overline{z} = 0$ $\sigma_y > 0$, $\sigma_z > 0$
Negative-mean non-market wealth	Case 5: $\overline{z} < 0$ $\sigma_y = 0$, $\sigma_z = 0$	Case 6: $\overline{z} < 0$ $\sigma_y > 0$, $\sigma_z = 0$	Case 7: $\overline{z} < 0$ $\sigma_y = 0$, $\sigma_z > 0$	Case 8: $\overline{z} < 0$ $\sigma_y > 0$, $\sigma_z > 0$
Positive-mean non-market wealth	Case 9: $\overline{z} > 0$ $\sigma_y = 0$, $\sigma_z = 0$	Case 10: $\overline{z} > 0$ $\sigma_y > 0$, $\sigma_z = 0$	Case 11: $\overline{z} > 0$ $\sigma_y = 0$, $\sigma_z > 0$	Case 12: $\overline{z} > 0$ $\sigma_y > 0$, $\sigma_z > 0$

1. $\overline{z}$ is the mean of non-market wealth

2. σ_z is the standard deviation of the non-market wealth at year 7

3. σ_y is the standard deviation of the multiplicative risk at year 7

Table 2: Portfolio Optimisation Example: Data

Expected Return			Horizon,	n	7 years
on Market		10%	Coefficient of		
Risk-free Rate		5%	Relative Risk aversion,	γ	1, 1.5, 2
			Expected Non-market		
Volatility of			Wealth,	$\overline{z}$	-20, 0, 30
Market Return,	σ_m	20%	Investible wealth,	x_0	100
			Standard deviation of		
			Non-market wealth	σ_z	0, 20, 30, 40
			Standard deviation of		
			Multiplicative risk	σ_y	0, 0.3

1. We assume that the market return follows a discrete binomial process, with a mean return of 10 % over each year. The volatility of the underlying continuous process, σ_m, is 20%.

2. The risk-free rate of interest is 5% on a discrete, annual basis.

3. In the right hand columns we show the investor characteristics. The horizon, when non-market wealth is realised is 7 years. The coefficient of relative risk aversion is γ =1, 1.5, or 2.

Table 3: Multiplicative and Non-Market Wealth Risk Effects on Stock Proportions

					Year 6: state							
Case	z	σ_ε	σ_y	Year 0	0	1	2	3	4	5	6	Derived Utility
1	0	0	0	78	78	78	78	78	78	78	78	CRRA
2	0	0	0.3	78	78	78	78	78	78	78	78	CRRA
3	0	30	0	66	77	76	74	70	64	53	37	DRRA
4	0	30	0.3	55	76	74	71	64	51	31	15	DRRA
5	-20	0	0	67	74	72	70	68	65	61	57	DRRA
6	-20	0	0.3	65	73	71	69	66	63	58	53	DRRA
7	-20	30	0	53	72	69	65	58	49	35	22	DRRA
8	-20	30	0.3	29	67	62	53	37	20	9	5	DRRA
9	30	0	0	94	83	85	87	92	98	109	127	IRRA
10	30	0	0.3	96	83	86	89	93	100	111	131	IRRA
11	30	30	0	85	82	84	85	86	87	84	73	? U-shape
12	30	30	0.3	83	83	84	85	86	85	77	57	? U-shape

1. All data is as shown in Table 2, with $\gamma = 1.5$. In column 1 the cases are numbered 1-12.

2. The state is indexed by the number of down-moves in the binomial process after 6 years.

3. $\overline{z}$ is the expected non-market wealth of the investor at time 0. σ_y is the standard deviation of the multiplicative risk. σ_z is the standard deviation of the non-market wealth.

4. Column 5 shows the stock proportion in the optimal portfolio in year 0.

5. Columns 6-12 show the stock proportion in the optimal portfolio in year 6 in the various states.

6. The derived utility is the utility for market wealth, $\nu(x)$. CRRA stands for constant relative risk aversion, DRRA for declining relative risk aversion, and IRRA for increasing relative risk aversion.

7. In Cases 11 and 12, "?" indicates that, although in these examples the investor acts like an agent with increasing or decreasing or U-Shaped relative risk aversion, this will not always be the case.

Table 4: Multiplicative and Non-Market Wealth Effects on Stock Proportions: Sensitivity Analysis

						Year 6: state							
Case	γ	$\overline{z}$	σ_z	σ_y	Year 0	0	1	2	3	4	5	6	Derived Utility
11	1.5	30	30	0	85	82	84	85	86	87	84	73	? U-shape
11a	1.5	30	20	0	90	82	84	86	89	93	98	102	? IRRA
11b	1.5	30	40	0	77	82	82	82	81	76	64	44	? DRRA
11c	2	30	30	0	63	62	63	63	63	63	61	57	? U-shape
11d	1	30	30	0	129	122	124	128	132	134	121	69	? U-shape
12	1.5	30	30	0.3	83	83	84	85	86	85	77	57	? U-shape
12a	1.5	30	20	0.3	91	83	85	87	90	94	98	100	? IRRA
12b	1.5	30	40	0.3	69	82	82	81	77	65	43	22	? DRRA
12c	2	30	30	0.3	61	63	63	63	63	61	57	49	? U-shape
12d	1	30	30	0.3	125	122	125	128	132	133	99	32	? U-shape

1. All data is as shown in Table 2.
2. The state is indexed by the number of down-moves in the binomial process after 6 years.
3. $\overline{z}$ is the expected non-market wealth of the investor at time 0. σ_y is the standard deviation of the multiplicative risk. σ_z is the standard deviation of

the non-market wealth.

4. Column 5 shows the stock proportion in the optimal portfolio in year 0.

5. Columns 6-12 show the stock proportion in the optimal portfolio in year 6 in the various states.

6. The derived utility is the utility for market wealth, $\nu(x)$. CRRA stands for constant relative risk aversion, DRRA for declining relative risk aversion, and IRRA for increasing relative risk aversion.

7. In Cases 11 and 12, "?" indicates that, although in these examples the investor acts like an agent with increasing or decreasing or U-Shaped relative risk aversion, this will not always be the case.

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Notes for Figures

In Figures 2-9, the trio $(\overline{z}, \sigma_z, \sigma_y)$ signifies the levels of expected non-market wealth, the risk of non-market wealth and the $\widetilde{y}$ risk.

1. Figure 1 depicts the derived RRA for $\gamma = 1.5$ and positive fixed non-market wealth $\overline{z} = 30$ and the derived RRA for risky non-market wealth with expectation 30. The probability distribution of z is : $z = 30 \pm 60$ with probability 1/16, $z = 30 \pm 30$ with probability 4/16, and $z = 30$ with probability 6/16. Also, the derived relative risk aversion is shown where y is 0.7 or 1.3 with equal probability. The graph shows the derived relative risk aversion a) in the absence of both additive and multiplicative background risks, b) in the absence of one of the background risks, c) in the presence of both additive and multiplicative background risks.

2. In Figure 2, we plot the logarithm of the total portfolio gross return over the 7 years against the logarithm of the market gross return, for the case where there is no non-market wealth risk and no $\widetilde{y}$ risk. The expected return on the market is 10% and the volatility of the market is 20%. The coefficient of relative risk aversion is $\gamma = 1.5$. The expected non-market wealth is $\overline{z} = 0$ in case 1, $\overline{z} = -20$ in case 5, and $\overline{z} = 30$ in case 9.

3. In Figure 3, we compare four different cases. In each case, the expected value of the non-market wealth is $\overline{z} = 0$, while the market data and the coefficient of relative risk aversion are the same as in the example in Figure 2. In case 1 we assume that there is no risk, i.e. neither non-market wealth risk nor $\widetilde{y}$ risk. In case 2, the risk of the non-market wealth is $\sigma_z = 0$ and the $\widetilde{y}$ risk $\sigma_y = 0.3$. In case 3, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk $\sigma_y = 0$. In case 4, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$.

4. In Figure 4, we compare four different scenarios. In each case, the expected value of the non-market wealth is $\overline{z} = -20$, while the market data and the coefficient of relative risk aversion are the same as in the example in Figure 2. In case 5 we assume that there is no risk, i.e. neither non-market wealth risk nor $\widetilde{y}$ risk. In case 6, the risk of the non-market wealth is $\sigma_z = 0$ and the $\widetilde{y}$ risk $\sigma_y = 0.3$. In case 7, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk $\sigma_y = 0$. In case 8, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$.

5. In Figure 5, we compare four different scenarios. In each case, the expected value of the non-market wealth is $\overline{z} = 30$, while the market data and the coefficient of relative risk aversion is the same as in the example in Figure 2.

In case 9 we assume that there is no risk, i.e. neither non-market wealth risk nor $\widetilde{y}$ risk. In case 10, the risk of the non-market wealth is $\sigma_z = 0$ and the $\widetilde{y}$ risk $\sigma_y = 0.3$. In case 11, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk $\sigma_y = 0$. In case 12, the risk of the non-market wealth is $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$.

6. In Figure 6, we illustrate the dynamic asset allocation strategy in the case where expected non-market wealth is $\overline{z} = 30$, $\sigma_z = 0$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$. This is the same as case 10 in Table 3. The line marked $0d$ shows the stock proportion in the state where there have been no down moves in the market return process by year t. The line marked $1d$ shows the stock proportion in the state where there has been one down move in the market return process by year t, and so on.

7. In Figure 7, we illustrate the dynamic asset allocation strategy in the case where expected non-market wealth is $\overline{z} = 0$, $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$. This is the same as case 4 in Table 3. The line marked $0d$ shows the stock proportion in the state where there have been no down moves in the market return process by year t. The line marked $1d$ shows the stock proportion in the state where there has been one down move in the market return process by year t, and so on.

8. In Figure 8, we illustrate the dynamic asset allocation strategy in the case where expected non-market wealth is $\overline{z} = 30$, $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0$. This is the same as case 11 in Table 3. The line marked $0d$ shows the stock proportion in the state where there have been no down moves in the market return process by year t. The line marked $1d$ shows the stock proportion in the state where there has been one down move in the market return process by year t, and so on.

9. In Figure 9, we illustrate the dynamic asset allocation strategy in the case where expected non-market wealth is $\overline{z} = 30$, $\sigma_z = 30$ and the $\widetilde{y}$ risk is $\sigma_y = 0.3$. This is the same as case 12 in Table 3. The line marked $0d$ shows the stock proportion in the state where there have been no down moves in the market return process by year t. The line marked $1d$ shows the stock proportion in the state where there has been one down move in the market return process by year t, and so on.

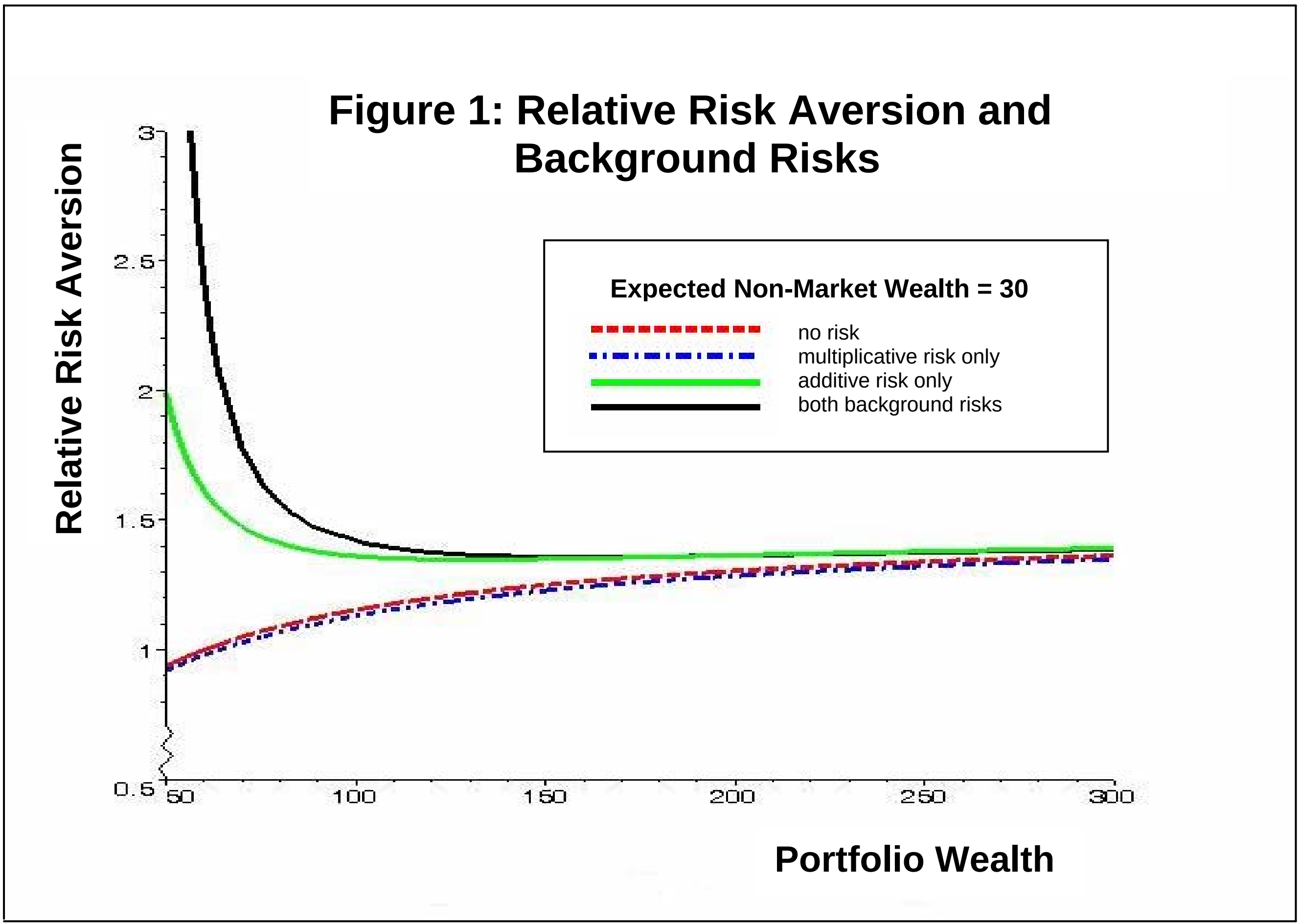


Figure 1: Relative Risk Aversion and Background Risks
Relative Risk Aversion
Expected Non-Market Wealth = 30
no risk
multiplicative risk only
additive risk only
both background risks
3
2.5
2
1.5
1
0.5
50
100
150
200
250
300
Portfolio Wealth

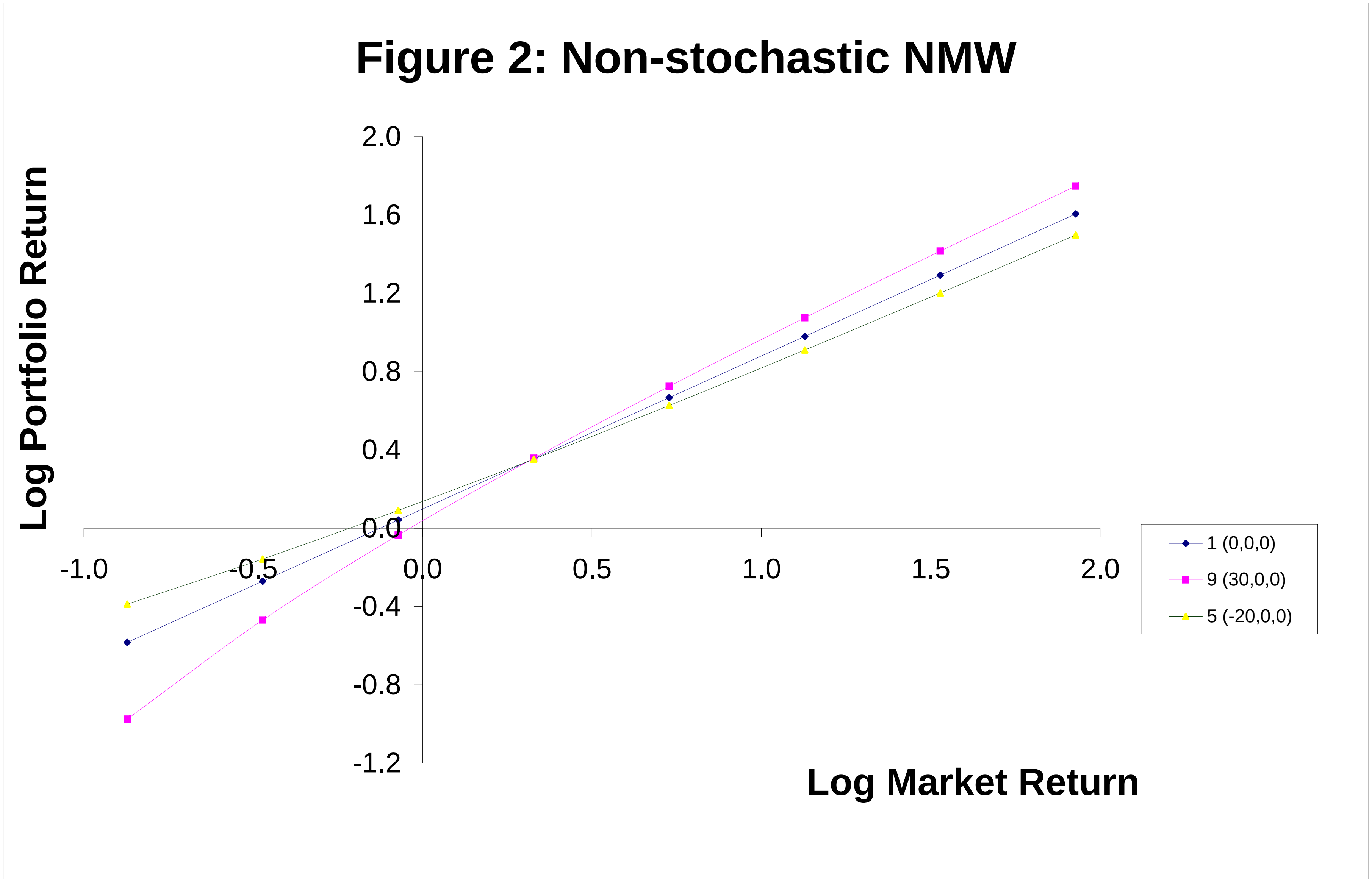
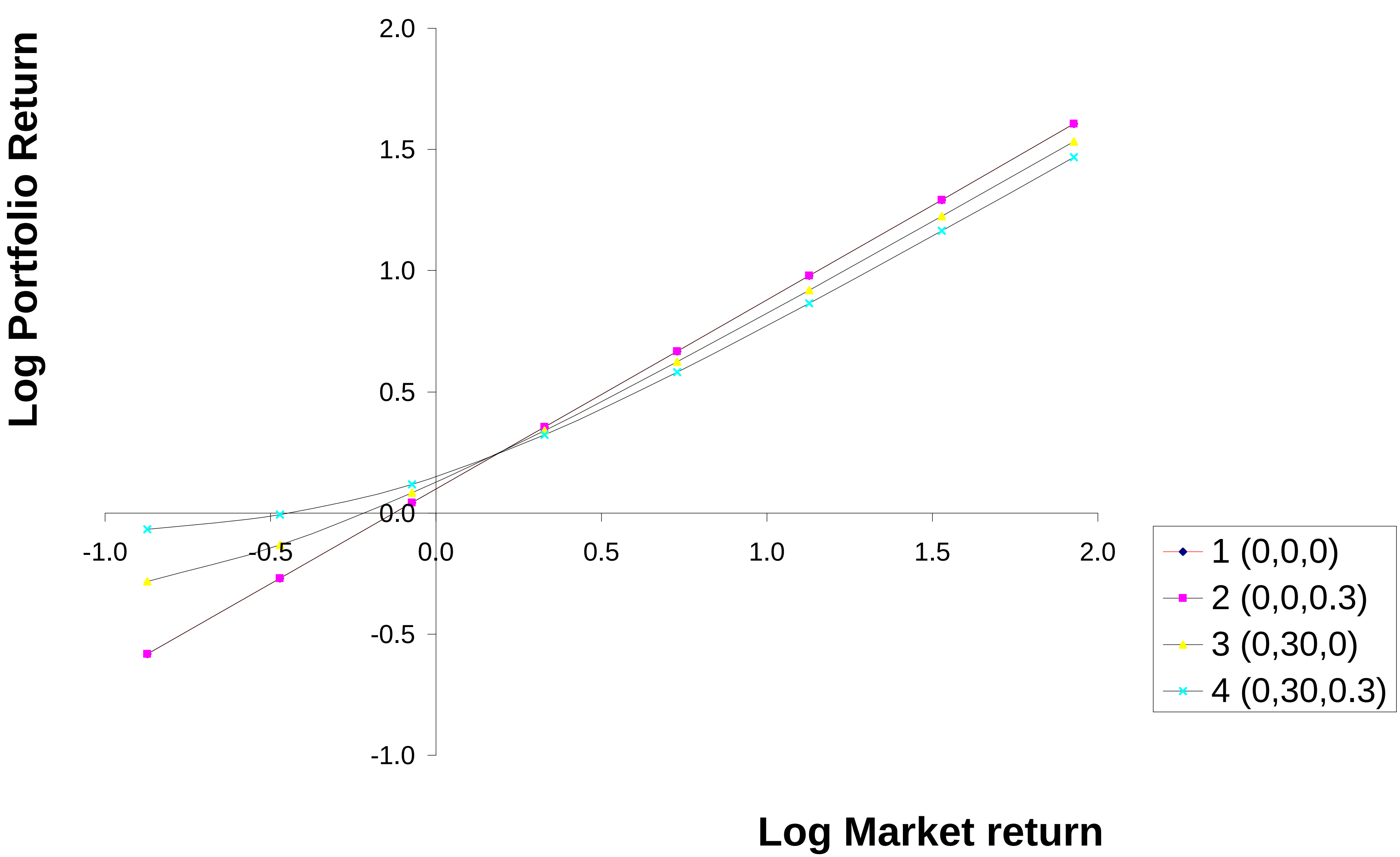


Figure 2: Non-stochastic NMW
Log Portfolio Return
Log Market Return
2.0
1.6
1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
1 (0,0,0)
9 (30,0,0)
5 (-20,0,0)

Figure 3: Expected NMW = 0



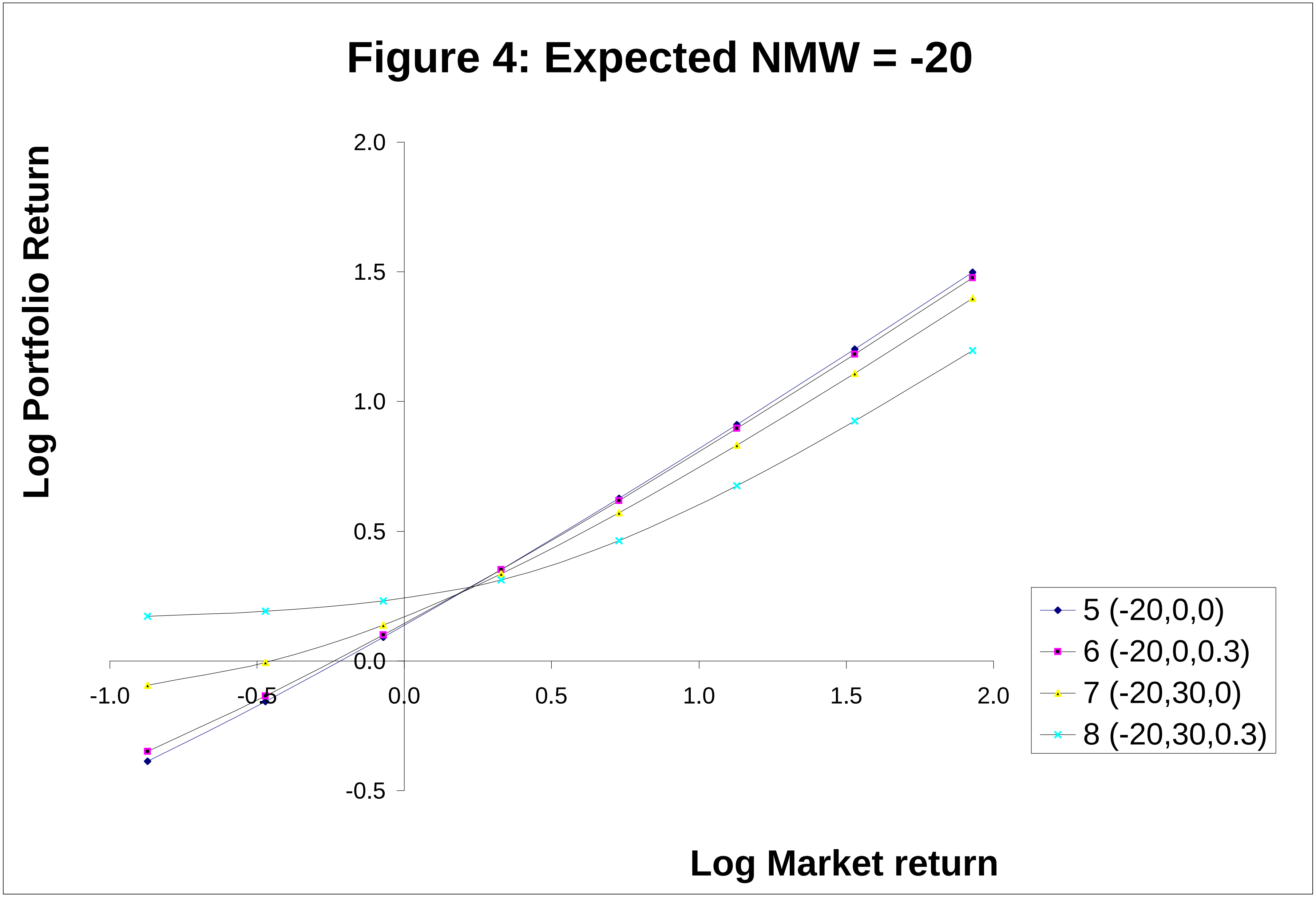


Figure 4: Expected NMW = -20
Log Portfolio Return
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
Log Market return
5 (-20,0,0)
6 (-20,0,0.3)
7 (-20,30,0)
8 (-20,30,0.3)

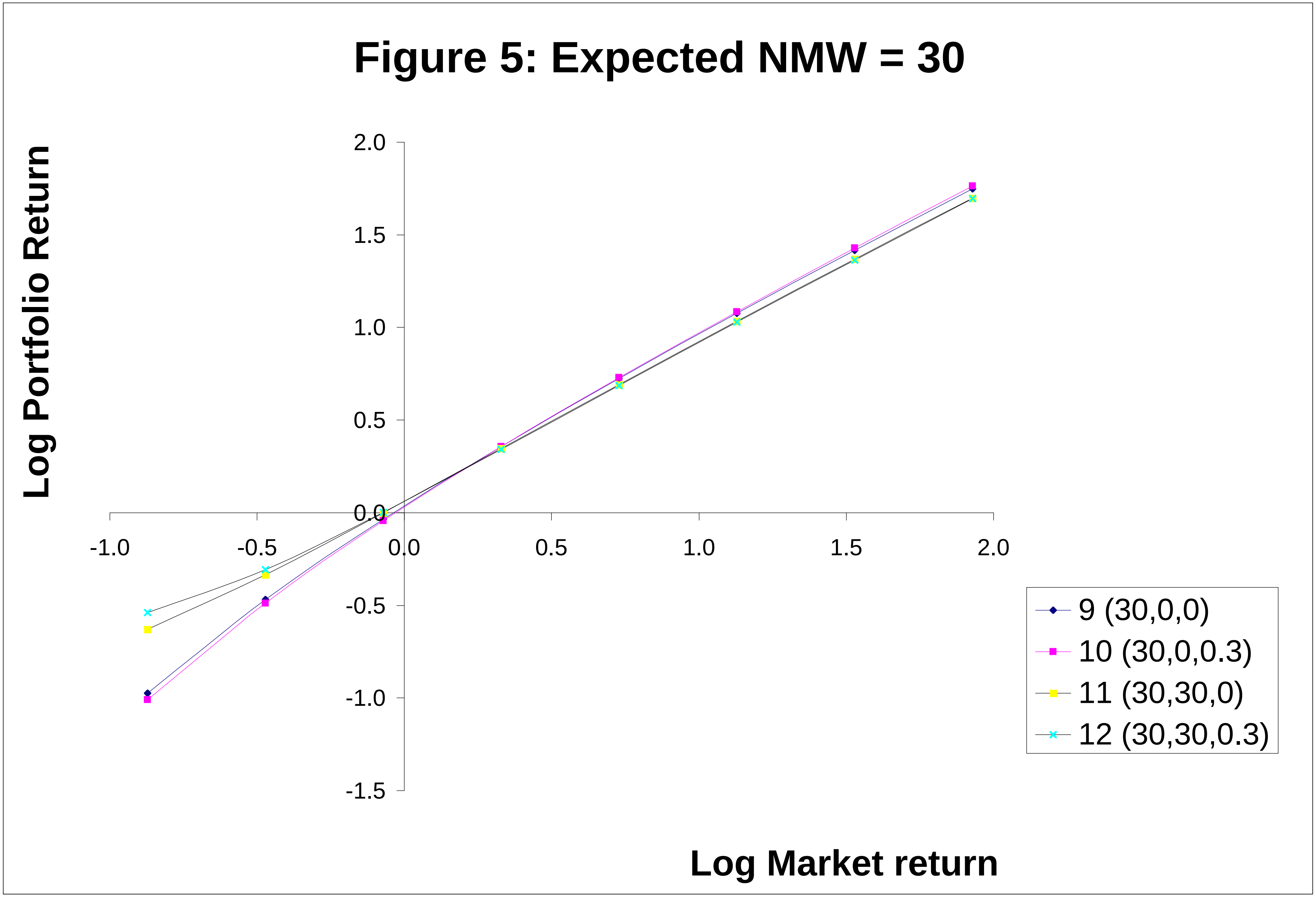


Figure 5: Expected NMW = 30
Log Portfolio Return
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
9 (30,0,0)
10 (30,0,0.3)
11 (30,30,0)
12 (30,30,0.3)
Log Market return

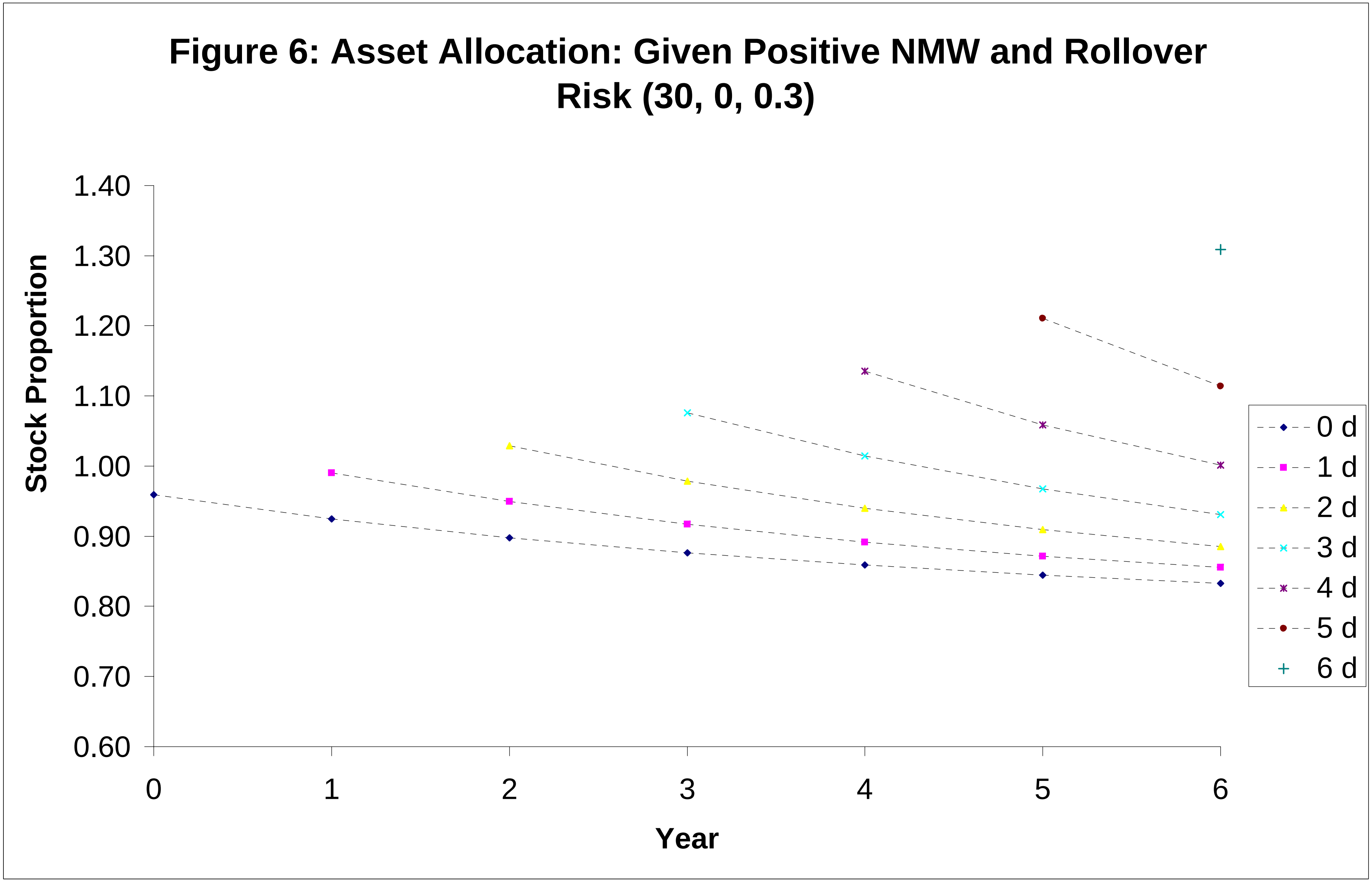


Figure 6: Asset Allocation: Given Positive NMW and Rollover Risk (30, 0, 0.3)
Stock Proportion
1.40
1.30
1.20
1.10
1.00
0.90
0.80
0.70
0.60
0
1
2
3
4
5
6
Year
0 d
1 d
2 d
3 d
4 d
5 d
6 d

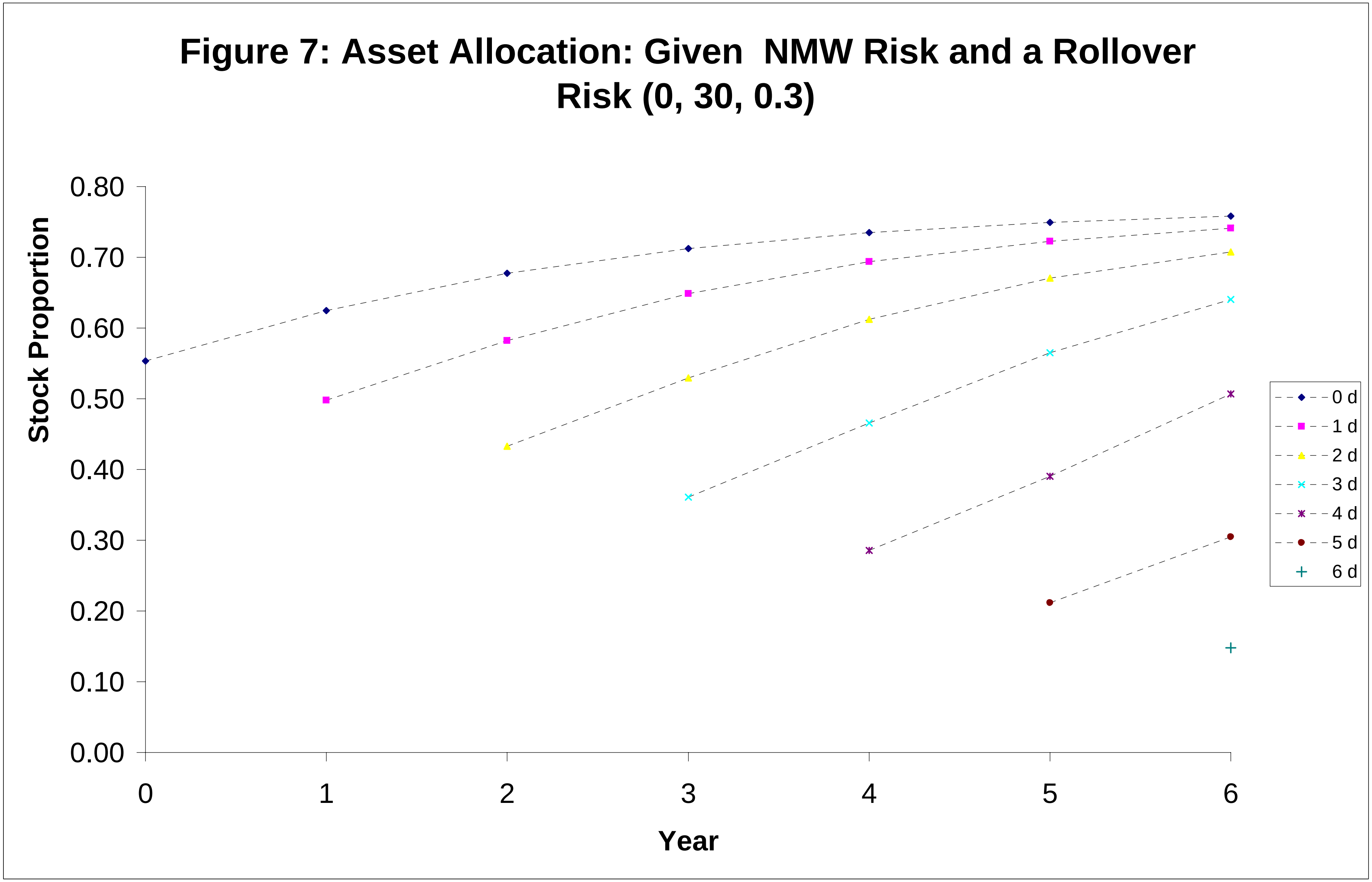


Figure 7: Asset Allocation: Given NMW Risk and a Rollover Risk (0, 30, 0.3)
Stock Proportion
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
Year
0 d
1 d
2 d
3 d
4 d
5 d
6 d

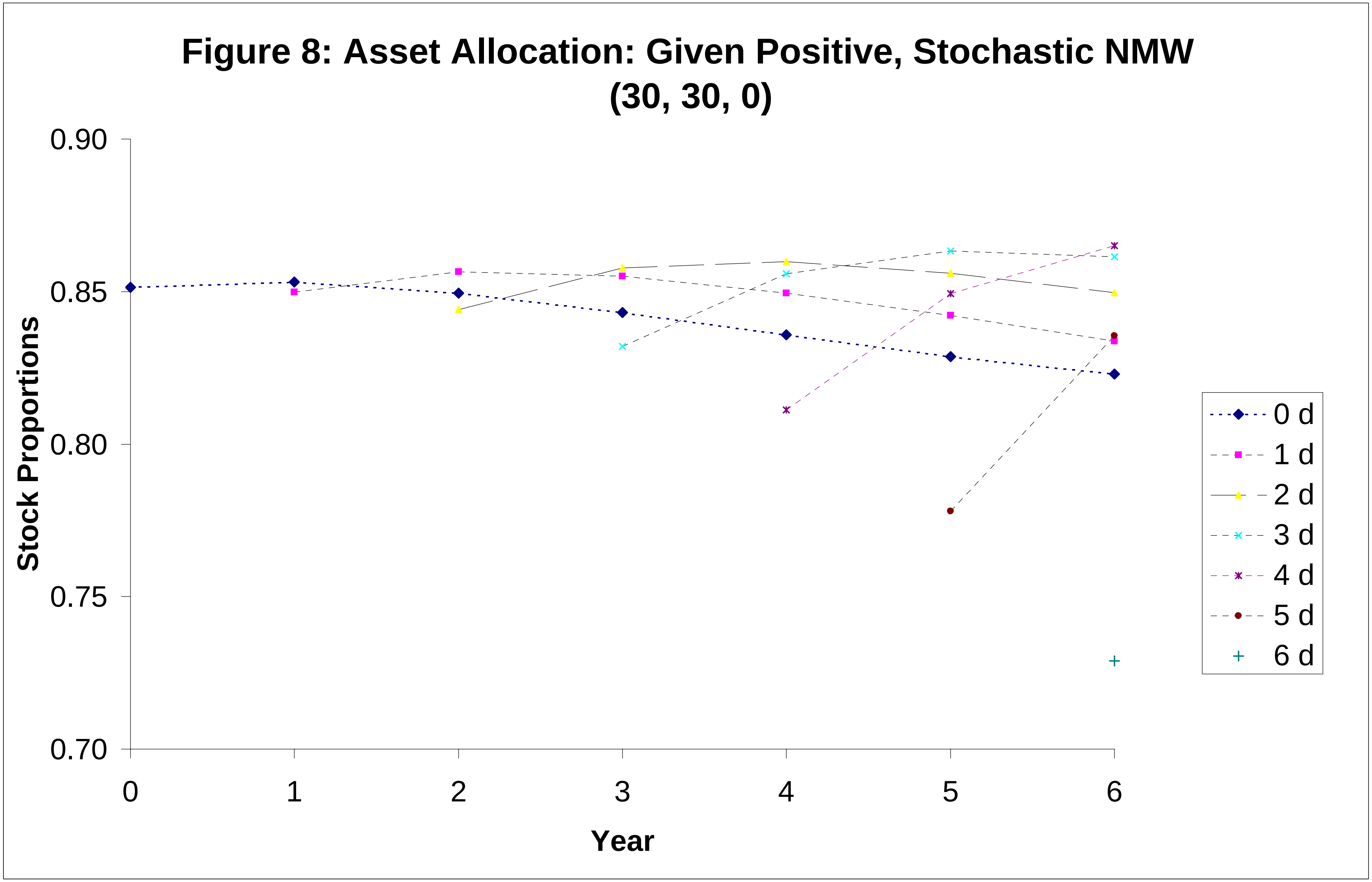


Figure 8: Asset Allocation: Given Positive, Stochastic NMW
(30, 30, 0)
Stock Proportions
0.90
0.85
0.80
0.75
0.70
0
1
2
3
4
5
6
Year
0 d
1 d
2 d
3 d
4 d
5 d
6 d

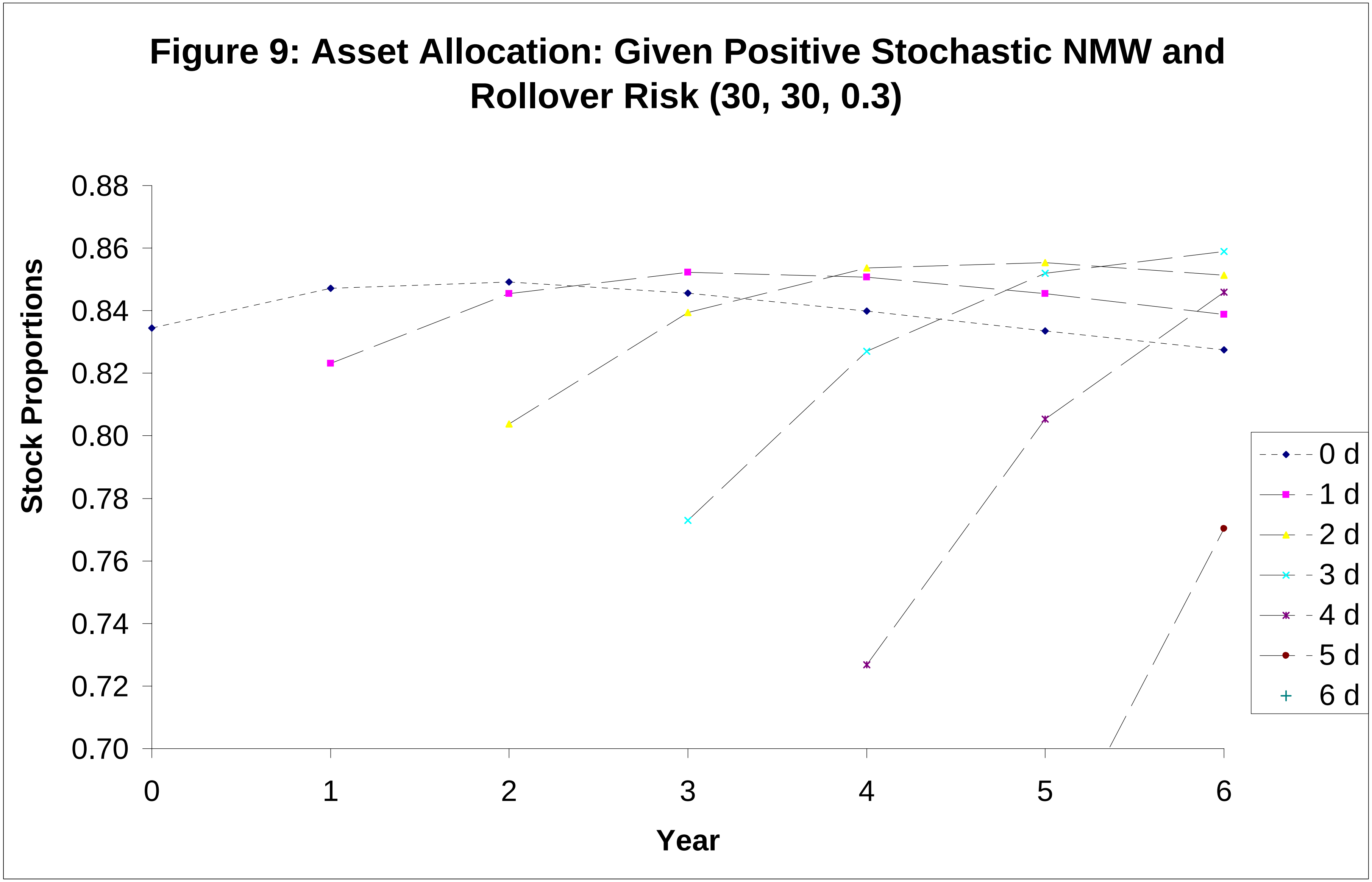


Figure 9: Asset Allocation: Given Positive Stochastic NMW and Rollover Risk (30, 30, 0.3)
Stock Proportions
0.88
0.86
0.84
0.82
0.80
0.78
0.76
0.74
0.72
0.70
0
1
2
3
4
5
6
Year
0 d
1 d
2 d
3 d
4 d
5 d
6 d