Cautiousness and Demand for Options

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Abstract

We characterize cautiousness, the rate of change of risk tolerance, using a simple portfolio problem in which agents invest in a stock, a risk-free bond and an option on the stock. We present three different characterizations by answering the following three questions: who buys options? who buys more options per share of the stock? who buys more put options than shares? These results show how cautiousness determines the demand for options.

Keywords: Cautiousness, demand for options, prudence, risk aversion.

JEL codes: D81, G11.

1 Introduction

Cautiousness, the rate of change of risk tolerance, was introduced by Wilson (1968) and used by Leland (1980) to study the demand for portfolio insurance. Cautiousness is equivalent to the ratio of prudence to risk aversion and hence is a measure of relative prudence. In his paper, Leland assumes a complete market and shows that 'other things being equal an agent with higher prudence relative to her risk aversion is more likely to buy portfolio insurance.'

In this paper we define an agent to be more cautious 'in the strong sense' if the minimum of her cautiousness exceeds the maximum of another agents' cautiousness. In order to characterize this concept we consider the simplest possible scenario where decision makers can buy or sell a single stock, a risk-free bond, and a convex derivative on the stock. As an example, the stock could be the market portfolio, and the derivative could be a call or a put option on the market portfolio. Using this simple portfolio problem, we present three different ways to characterize cautiousness by answering the following three questions: who buys options? who buys more options per share of the stock? who buys more put options than shares?

To some extent, our analysis provides a direct extension of the results in Arrow (1965) and Pratt (1964). Arrow and Pratt show that given the choice between investing in a positive excess-return risky asset and a risk-free asset, an agent will always invest more in the risky asset than another agent *if and only if* she has lower absolute risk aversion. Investment in the risky asset characterizes absolute risk aversion. We show that, given the additional choice of investing in an option, an agent has higher cautiousness *if and only if* she is always more likely to buy options. Also, she always demands a higher ratio of options to shares *if and only if* she has higher cautiousness. Thirdly, assuming the option is a put option, she buys more options than shares *if and only if* she has higher cautiousness. Hence investment in the option characterizes cautiousness.

The scenario where decision makers can buy or sell a single stock, a risk-free bond, and a convex derivative on the stock is chosen because it is the simplest way to characterize cautiousness. Of course, in practice investors can invest in a range of stocks and derivatives with various strike prices, however if we show that cautiousness determines demand in our simple scenario then it clearly must influence choice in more complex scenarios.

Our results here are closely related to the previous work of Leland (1980) and Brennan and Solanki (1981). As mentioned above Leland (Proposition 1) shows that an agent with higher cautiousness is more likely to buy portfolio insurance. Brennan and Solanki obtain a similar result in a lognormal model where a risk-neutral valuation relationship holds for the valuation of the derivative. These two studies give a first attempt to characterize cautiousness by the demand for portfolio insurance (options), albeit using the convexity of an investor's optimal terminal payoff function as a proxy of his demand for portfolio insurance (options), in a market with a complete set of options. It is obvious that when there is more than one option, it is sometimes difficult to determine who is an option buyer and who is an option seller as an investor may buy some options and sell the others; thus this may lead to a problem in the above characterization of cautiousness. This problem is explained in details by Hara, Huang and Kuzmics (hereafter HHK) (2007). They show in their Theorem 18 that in an economy where consumers have different constant cautiousness, all consumers have terminal payoff functions which are initially convex and eventually concave except for those who are the most or the least cautious. They conclude that the results of Leland (1980) and Brennan and Solanki (1981) "are valid in a two-consumer economy, but do not generalize to an economy with a large number of consumers with diverse levels of relative risk aversion".[1]

In a related paper on the effect of background risk Franke, Stapleton and Subrahmanyam (hereafter FSS) (1998) also show that a convex payoff is optimal in a model where background risk increases the cautiousness of an investor with a HARA class utility function. HHK (2011) extend the above discussion about the effect of background risk on cautiousness to a more general class of utility functions.

[1] See the discussions of Theorem 18 in HHK (2007).

Cautiousness has also been used in analyzing other problems. For example, Gollier (2001) discusses how an investors' cautiousness is related to the local convexity of his consumption rule. He shows that an agent's consumption rule is locally convex (concave) if his cautiousness is larger (smaller) than the weighted average cautiousness.[2] In an earlier related study, Carroll and Kimball (1996) investigate the effect of uncertainty on the curvature of investors' consumption rules by examining its effect on their cautiousness. They show that if investors have HARA class utility functions, then uncertainty will increase their cautiousness, which leads to concave optimal consumption rules. HHK (2007) show how heterogeneity in cautiousness affects consumers' portfolio strategies and the representative consumer's preference. Gollier (2007) finds that cautiousness helps to explain the aggregation of heterogeneous beliefs. Gollier (2008) further shows that cautiousness plays an important role in understanding saving and portfolio choices with predictable changes in asset returns.

The structure of this paper is as follows. In Section 2, we introduce the concept of being more cautious and the expected utility maximizing model which underlies our analysis. In Section 3, we characterize cautiousness by answering the question who buys options. In Section 4, we look at the question of relative option demand: who buys more options per share of the stock? Then, in Section 5, we ask the question: who buys more options than shares? In Section 6, we give some numerical examples to illustrate the main results. In Section 7, we present two applications of the main results: (1) how does cautiousness determine the demand for stocks v.s. corporate bonds? and (2) how does background risk affect the demand for options? The final section concludes the paper.

[2]See Gollier (2001) page 207, Proposition 52.

2 The Model

2.1 The Concept of Cautiousness

Cautiousness was first defined by Wilson (1968) based on another risk preference measure, the Pratt-Arrow risk aversion. Given a utility function $u(w)$, the function $R(w) = -u''(w)/u'(w)$ is the well-known Pratt-Arrow risk aversion, a concept developed by Pratt (1964) and Arrow (1965). The inverse of this function, $T(w) = -u'(w)/u''(w)$ is called risk tolerance. Cautiousness is defined as the rate of change of risk tolerance, i.e., cautiousness is the function $C(w) = T'(w)$.[3]

Cautiousness is also closely related to another well-known risk preference measure, the measure of prudence. Prudence is defined by Kimball (1990) as $P(w) = -u'''(w)/u''(w)$. We have

$$(\frac{1}{R(w)})' = -\frac{(\ln R(w))'}{R(w)} = -\frac{(\ln(-u''(w)))' - (\ln u'(w))'}{R(w)} = \frac{P(w)}{R(w)} - 1.$$

Thus cautiousness is equivalent to the ratio of prudence to risk aversion minus one.

Now we define a key concept in this paper.

Definition 1 *Investor i is said to be more cautious than investor j if there exists a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$, where $C_i(w)$ and $C_j(v)$ are the cautiousness measures of investors i and j respectively.*[4]

It is straightforward that the condition in the definition is equivalent to $\inf_w C_i(w) \geq \sup_v C_j(v)$. The above concept gives an ordering of utility functions in terms of their cautiousness. Since HARA class utility functions have constant cautiousness they can be ordered perfectly in this way.

[3]Throughout the paper, we use R and C to denote risk aversion and cautiousness respectively.

[4]Throughout the paper, when we say for all w and v, $C_i(w) \geq C \geq C_j(v)$, we mean for all w and v in the natural domains of $u_i(w)$ and $u_j(v)$ respectively.

2.2 A Simple Portfolio Problem

Assume a two-date economy with starting time 0 and ending time 1. Assume there is a risk-free bond traded in the market; the risk-free interest rate is denoted by r. Assume there is a stock available in the market whose prices at time 0 and 1 are denoted by S_0 and S respectively. We assume that the distribution of the stock price S is continuous and its support, denoted by I, is a bounded subinterval of $[0, +\infty)$. Although we assume that the stock price follows a continuous distribution, the results obtained in this paper can easily be extended to the discrete case.[5]

Assume there is a convex derivative written on the stock that is traded in the market. A convex derivative is a derivative with a convex payoff function. We assume that the convex derivative's payoff function is piecewise differentiable with a finite number of non-differentiable points and its derivative is bounded in the entire support. To further reduce technicality, we assume that the convex derivative's payoff function is twice differentiable in every differentiable interval.[6] Moreover, to ensure that the convex derivative will not degenerate to a fraction of the stock, we assume that its payoff function is strictly convex for at least one interior point (S^*) of the support. Note that a call or a put option with strike price K inside the support interval I, is an example of such a convex derivative.

Denote the payoff of the derivative at time 1 by $a(S)$. Note since $a(S)$ is a convex function of S, $a(S)$ is continuous in the interior of the support. Denote the price of the derivative at time 0 by a_0. The interest rate and the current prices of the stock and the derivative are exogenous.

We stress here that we do not assume all investors are rational utility-maximizers. We only assume there are some investors who are rational expected-utility-maximizers whose behaviors in the option market are the subject of this

[5]The boundedness of the support I is not required for Statement 1 in Theorem 1 to imply Statement 2, which can clearly be seen from the proof of the theorem.

[6]This assumption can be relaxed. The proof of Theorem 1 for the case where the convex derivative's payoff function is piecewise continuous differentiable is available on request.

research. These investors are indexed by $i = 1, 2, ...$; and they are all price-takers. Investor i's preference is represented by utility function $u_i(x)$. At time 0 he has initial wealth w_{0i}. Assume that at time 0 he buys x_i shares of the stock and y_i units of the derivative, and invests the rest of his wealth $(w_{0i} - x_i S_0 - y_i a_0)$ in the money market. Denote investor i's wealth at time 1 by $w_i(S; x_i, y_i)$. We have

$$w_i(S; x_i, y_i) = (w_{0i} - x_i S_0 - y_i a_0)(1 + r) + x_i S + y_i a(S). \tag{1}$$

For brevity we will often write $w_i(S; x_i, y_i)$ simply as $w_i(S)$. Note as $a(S)$ is continuous and piecewise twice differentiable, $w_i(S)$ is also continuous and piecewise twice differentiable.

Investor i maximizes the expected utility of his time 1 wealth $w_i(S)$. That is,

$$\max_{x_i, y_i} E u_i(w_i(S)). \tag{2}$$

We obtain the first order conditions:

$$E[u_i'(w_i(S))(S - (1 + r)S_0)] = 0, \quad \text{and} \quad E[u_i'(w_i(S))(a(S) - (1 + r)a_0)] = 0,$$

which can be written as

$$\frac{E[u_i'(w_i(S))S]}{Eu_i'(w_i(S))} = (1 + r)S_0, \quad \text{and} \quad \frac{E[u_i'(w_i(S))a(S)]}{Eu_i'(w_i(S))} = (1 + r)a_0. \tag{3}$$

The solution, (x_i, y_i), depends on the utility function, on the prices (S_0, a_0), and on the initial wealth of the investor given the interest rate r and the distribution of the stock price. We assume that all utility functions are strictly increasing, strictly concave, and three times continuously differentiable. The strict concavity of the utility functions guarantees that the second order condition for the expected utility maximization problem is always satisfied and a solution to (3) is a global maximum which is unique.

Before we proceed to analyze the optimal solution, we first introduce some notation. Let $\phi_i(S) \equiv u_i'(w_i(S))/Eu_i'(w_i(S))$. Then (3) can be written as

$$E[\phi_i(w_i(S))S] = (1 + r)S_0, \quad \text{and} \quad E[\phi_i(w_i(S))a(S)] = (1 + r)a_0. \tag{4}$$

Thus $\phi_i(S)$ can be regarded as investor i's individual pricing kernel, which she uses to price the stock and the derivative. As the investor has to take the market prices as given, her individual pricing kernel must price the stock and the derivative correctly; that is, an individual pricing kernel must be admissible. As is well known, this individual pricing kernel is the Radon-Nikodym derivative of an equivalent Martingale measure with respect to the true probability measure. Thus there must exist an equivalent Martingale measure $Q_i(S)$ such that $\phi_i(S) = \frac{dQ(S)}{dP(S)}$, where $P(S)$ denotes the true probability measure.

Note that if the market is complete, then there is a unique equivalent Martingale measure, which leads to a unique admissible pricing kernel: the market pricing kernel (or the representative investor's pricing kernel); thus all individual pricing kernels must be equal to this pricing kernel. In this case, if we are given the market pricing kernel, as in Leland (1980) and Brennan and Solanki (1981), it will be straightforward to obtain the relation between an investor's demand for the derivative and her utility function.

However, when the market is incomplete, admissible pricing kernels are not unique. An individual pricing kernel can be any one of the many admissible pricing kernels, and it cannot be known unless we solve the investor's portfolio problem. Thus, in this case, it becomes more difficult to obtain the relation between an investor's demand for the derivative and her risk preferences.

To understand the characteristics of admissible pricing kernels, we may note that as $w_i(S)$ is continuous and piecewise twice differentiable and $u_i(w)$ is three times differentiable, $\phi_i(S) \equiv u_i'(w_i(S))/Eu_i'(w_i(S))$ is also continuous and piecewise twice differentiable. In every differentiable interval, let $\delta_i(S)$ denote the negative derivative of the logarithm of investor i's individual pricing kernel, i.e., $\delta_i(S) \equiv -\phi_i'(S)/\phi_i(S)$. From the definitions of $\phi_i(S)$ we have

$$\delta_i(S) = R_i(w_i(S))w_i'(S). \tag{5}$$

In every such interval, as $w_i(S)$ is twice differentiable and $R_i(w)$ is differentiable, $\delta_i(S)$ is also differentiable; then it is bounded in any bounded subinterval. Thus $\delta_i(S)$ is well defined except for a finite number of points, at which the convex derivative's payoff function is not differentiable, and it is bounded in any bounded subinterval of the interior of the support. As is well known, a bounded and almost everywhere continuous function is Riemann integrable; hence $\delta_i(S)$ is Riemann integrable. It follows that for any S and a, two points in the interior of the support, we have

$$\ln \frac{\phi_i(S)}{\phi_i(a)} = -\int_a^S \delta_i(x)dx. \tag{6}$$

We now present a lemma which shows some characteristics of admissible pricing kernels.

Lemma 1 *Assume two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ are continuous and piecewise continuously differentiable. If they both price the stock correctly, then the following statements are true.*

1. *The two pricing kernels must intersect at least twice.*
2. *$\delta_i(S)$ and $\delta_j(S)$ must intersect at least once.*
3. *Assume $\delta_i(S)$ crosses $\delta_j(S)$ once at $S = b$. Then $\phi_i(b) \neq \phi_j(b)$.*
4. *If $\delta_i(S)$ crosses $\delta_j(S)$ once, then $\phi_i(S)$ crosses $\phi_j(S)$ twice.*

Proof: See Appendix A.

This lemma will be used repeatedly later in the proofs of our main results in this paper.

3 Who Buys Options?

In this section we will characterize the concept of cautiousness and show that cautiousness is a measure of an investor's motive to buy options. In the model above, where investors allocate their wealth between a stock, a convex derivative on the stock and a risk free bond, suppose there are two investors: i who buys y_i derivatives and j who buys y_j derivatives. We establish conditions under which the $sign(y_i) \geq sign(y_j)$, i.e. i is more likely to buy options than j. We now present our first main result.

Theorem 1 *The following two statements are equivalent.*

1. *Investor i is more cautious than investor j.*

2. *Given any initial wealth, stock price, and derivative price such that there is a solution to problem (3) for both investors i and j, investor j holds a (strictly) positive position in the derivative only if investor i does so, i.e., $y_j \geq (>) 0$ only if $y_i \geq (>) 0$.*

The proof of the above theorem is quite complicated, and its details can be found in Appendix D. But to help readers understand the proof, we now explain the basic idea used to prove that Statement 1 implies Statement 2. The proof is by contradiction. Suppose that Statement 1 does not imply Statement 2, i.e., although investor i is more cautious than investor j, it happens that investor i holds a negative position in the derivative while investor j holds a strictly positive position in the derivative, or investor i holds a strictly negative position in the derivative while investor j holds a positive position in the derivative. We need only to show that in either case we have a contradiction. The key is to prove that in either case " the two investors' individual pricing kernels can cross each other at most twice." We may call this the "insufficient crossing property."

Note that if insufficient crossing happens, then it can be shown that the two pricing kernels give strictly different prices for the convex derivative.[7] This implies that they can not both price the derivative correctly, which leads to a contradiction. From this we can conclude that Statement 1 implies Statement 2.

To prove the insufficient crossing property, we need only prove that the derivative of the logarithm of investor i's individual pricing kernel crosses the derivative of the logarithm of investor j's individual pricing kernel at most once. But as the payoff function of the derivative is only *piecewise* (twice) differentiable, the investors' individual pricing kernels are only *piecewise* (twice) differentiable. This causes some complexity to the proof.

[7] Franke, Stapleton and Subrahmanyam (1999) show that when two pricing kernels cross each other twice, the more convex pricing kernel of the two gives lower prices for all options.

If however the payoff function of the derivative were twice differentiable in the entire support, the situation would be much easier. To illustrate the basic idea of the proof, we now explain the proof for this special case under the assumption that utility functions are HARA class. Note as utility functions are HARA class, we have for all w and v, $C_i(w) \equiv C_i > C_j \equiv C_j(v)$.

As is explained in the last section, $\delta_i(S)$ denotes the negative derivative of the logarithm of investor i's individual pricing kernel, i.e., $\delta_i(S) \equiv -\phi_i'(S)/\phi_i(S)$. It is straightforward that if the derivative of $1/\delta_i(S)$ can not cross the derivative of $1/\delta_j(S)$, $\delta_i(S)$ can cross $\delta_j(S)$ at most once. To prove the derivative of $1/\delta_i(S)$ cannot cross the derivative of $1/\delta_j(S)$, applying (5) and differentiating $\delta_i(S)$ with respect to S, we obtain

$$\delta_i'(S) = -C_i(w_i(S))\delta_i^2(S) + R_i(w_i(S))w_i''(S). \tag{7}$$

If $\delta_i(S)$ is non-zero, noting that $C_i(w_i(S)) \equiv C_i$, we can rewrite it as

$$(\frac{1}{\delta_i(S)})' = C_i - \frac{R_i(w_i(S))w_i''(S)}{\delta_i^2(S)}.$$

Note that if investor i holds a negative position in the derivative while investor j holds a strictly positive position in the derivative or investor i holds a strictly negative position in the derivative while investor j holds a positive position in the derivative, then $w_i''(S) \leq 0 \leq w_j''(S)$. Also note that as all utility functions are strictly increasing and strictly concave guarantees, risk aversion $(R_i(w_i(S)))$ is always strictly positive. As we have $C_i > C_j$, it follows that $(\frac{1}{\delta_i(S)})' > (\frac{1}{\delta_j(S)})'$. This implies that $\delta_i(S)$ can cross $\delta_j(S)$ at most once and they are never equal to each other except for one point. As $\delta_i(S)$ $(\delta_j(S))$ is the derivative of the logarithm of investor i's (j's) individual pricing kernel, this further implies that investor i's individual pricing kernel can cross investor j's individual pricing kernel at most twice and they are never equal to each other except for two points. But as the two individual pricing kernels both price the underlying stock correctly, they must cross each other at least twice.[8] Thus they must cross each other exactly twice and they are never equal to each other except for

[8]See Statement 1 of Lemma 1.

two points. Then from Franke, Stapleton and Subrahmanyam (1999) investor i's individual pricing kernel gives strictly higher prices to all convex derivatives than investor j's individual pricing kernel.[9] As both individual pricing kernels are admissible, this causes a contradiction.

When the payoff function of the derivative is not twice differentiable in the entire support but is piecewise (twice) differentiable, and when the utility functions are not HARA class, the proof is similar in spirit but much more technical, as is shown in Appendix D.

The above theorem gives an ordering of utility functions in terms of the motive to buy options. This ordering is perfect for HARA class utility function as they all have constant cautiousness. Thus if investor i and j have constant cautiousness C_i and C_j, i.e., they have HARA class utility functions, and $C_i > C_j$, then investor i will have a stronger motive to buy options. Moreover, as stated earlier, for an exponential utility function, cautiousness is zero while any utility function which displays decreasing absolute risk aversion has positive cautiousness and any utility function which displays increasing absolute risk aversion has negative cautiousness. Thus according to the above theorem, any investor who has decreasing (increasing) absolute risk aversion always has a stronger (weaker) motive to buy options than an investor with an exponential utility function.

Furthermore, the theorem also implies the role of prudence in explaining the demand for options. According to Leland (1968) and Kimball (1990), an investor is prudent (imprudent) if his utility function has a positive (negative) third derivative. Consider the situation when one investor is prudent while another is imprudent. In this case, as cautiousness can be written as $C(w) = u'''(w)u'(w)/u''^2(w) - 1$, the first investor's cautiousness is larger than negative unity while the second investor's cautiousness is smaller than negative unity. According to Theorem 1, this implies that the first investor has a stronger motive to buy the convex derivative. Thus a prudent investor has a stronger motive to buy options than an imprudent investor.

[9] See also the proof of Lemma 7 in Appendix D.

4 Who Buys More Options Per Share of the Stock?

In the last section we characterized cautiousness by answering the question who buys options. Then, in Section 5, In this section we characterize cautiousness by answering another interesting question: who buys more options per share of the stock? Recall that in the model in Section 2, x_i is the number of shares of stock demanded by investor i. Then y_i/x_i is the relative amount of options demanded. We now establish conditions under which the ratio y_i/x_i exceeds y_j/x_j. We present the following result.

Theorem 2 *The following two conditions are equivalent.*

1. *Investor i is more cautious than investor j.*
2. *Given any initial wealth, stock price, and derivative price such that there is a solution to problem (3) for both investors i and j, if $x_iS + y_ia(S)$ is strictly monotone, then $x_j > (<)0$ implies $\frac{y_i}{x_i} \geq (\leq)\frac{y_j}{x_j}$; if $x_jS + y_ja(S)$ is strictly monotone, then $x_i > (<)0$ implies $\frac{y_i}{x_i} \geq (\leq)\frac{y_j}{x_j}$.*

Proof: See Appendix B.

The condition that $x_iS + y_ia(S)$ or $x_jS + y_ja(S)$ is monotone is necessary for the conclusion in the theorem; this is shown by contradiction in Section 6 by using some numerical examples.[10]

Also note that this condition is equivalent to that investor i's terminal wealth is a monotone function of the underlying stock price S. To understand this condition, consider the case where you have bought some units of a stock index. If you set up a normal portfolio insurance strategy by using an option on the index, your terminal wealth will be a monotone increasing function of the index unless you over-insure your stock index. Thus if you do not over-insure your stock index, the condition in the theorem will be satisfied.

[10]See the discussions at the end of Section 6.

Consider another case where you have written some put options on a stock. If you buy some shares of the underlying stock to cover these put options, your terminal wealth will be a monotone decreasing function of the stock price unless you over-cover your put options. Thus if you do not over-cover your put options, the condition in the theorem will be satisfied.

We may relate this result to a central result in Leland (1980) and Brennan and Solanki (1981) that if investor i is more cautious than investor j then $\frac{f_i''(x)}{f_i'(x)} \geq \frac{f_j''(x)}{f_j'(x)}$, where $f_i(x)$ and $f_j(x)$ are investor i and j's optimal terminal payoff functions and x is the aggregate wealth.[11] Now think of the aggregate wealth as a stock. If we interpret $f_i'(x)$ and $f_i''(x)$ as investor i's shares of the stock and his position in options on the stock respectively, then the connection of these two results is straightforward. However, we must note that the above theorem requires that $x_iS+y_ia(S)$ or $x_jS+y_ja(S)$ is monotone, and it is shown in Section 6 that if this condition is violated, then the result in the theorem will not hold. This shows that the conclusions reached by Leland (1980) and Brennan and Solanki (1981) in a market with a complete set of contingent claims cannot be simply extended to an incomplete market.

5 Who Buys More Put Options than Shares?

In Section 4 we characterized cautiousness by answering the question who buys options. Then, in Section 5, we characterized cautiousness by answering the question who buys more options per share of the stock. Now, in this section we characterize cautiousness by answering a third question: who buys more put options than shares? Using the notation introduced in Section 3, we ask when is $y_i \geq x_i$ and when is $y_i \leq x_i$. The relevance of this question is that the answer can determine, for example, the conditions under which an agent may under (over) insure a stock portfolio.

Unlike the other parts of the paper, in this section we focus only on options,

[11] See also the discussions about the result in Leland (1980) and Brennan and Solanki (1981) on page 657 in HHK (2007).

a special type of convex derivatives whose payoffs are two-segment and piecewise linear. It is obvious that an option's payoff function is strictly convex at only one point, the strike price, which connects the two segments. According to the definition of a convex derivative, we require the strike price to be an interior point of the stock price distribution's support.

Moreover, as the portfolio problem with a call option can be transformed into the portfolio problem with a put option by using the put-call parity, without loss of generality, we assume the derivative is a put option.

We have the following result.

Theorem 3 *If the convex derivative is a put option, then the following three conditions are equivalent.*

1. *Investor i is more cautious than investor j.*
2. *Given any initial wealth, stock price, and derivative price such that there is a solution to problem (3) for both investors i and j, if $x_i - y_i \geq (>)0$, then $x_j - y_j \geq (>)0$.*
3. *Given any initial wealth, stock price, and derivative price such that there is a solution to problem (3) for both investors i and j, if $x_j \geq (>)0$, then $x_i \geq (>)0$.*

Proof: See Appendix C.

Note as the slope of the first segment of a put option's payoff function is minus one, $x_i - y_i$ is the slope of the first segment of investor i's terminal wealth function $w_i(S)$. Similarly, x_i is the slope of the second segment of investor i's terminal wealth function $w_i(S)$.

Statement 2 of the theorem states that investor i buys (strictly) more shares than put options only if investor j does so or investor j buys (strictly) more put options than shares only if investor i does so. Thus the theorem tells us that a more cautious investor tends to buy more put options than shares.

Now consider some special cases. It is well known that if $x_i > 0$ and $y_i = x_i$, investor i is said to hold a fully-insured portfolio. If $x_i > 0$ and $y_i \in (0, x_i)$, investor i is said to hold a partially-insured portfolio. If $x_i > 0$ and $y_i > x_i$, investor i is said to hold an over-insured portfolio. From the above theorem we have the following result.

Corollary 1 *Assume investor i is more cautious than investor j. Then investor j buys an over-insured portfolio only if investor i does so, and investor i sells an over-insured portfolio only if investor j does so.*

Proof: If investor j buys an over-insured portfolio, then we have $x_j > 0$ and $x_j - y_j < 0$. Applying Theorem 3, we obtain $x_i > 0$ and $x_i - y_i < 0$, i.e., investor i buys an over-insured portfolio. Similarly, we can show that investor i sells an over-insured portfolio only if investor j sells an over-insured portfolio. Q.E.D.

6 Numerical Examples of Option Demand

In this section we present some numerical examples. These are designed to illustrate the conclusions of the Theorems established above. Table 1 shows optimal stock and option demands given three different sets of (S_0, a_0) prices. In part a), $S_0 = 84$ and $a_0 = 3.00$. Marginal utility is of the HARA class with

$$u'(w) = (w + \alpha)^{-\gamma}.$$

As discussed above, for this utility function cautiousness is a constant with $C(w) = 1/\gamma$ and absolute risk aversion $R(w) = \gamma/(\alpha + w)$. Cautiousness is shown for four different levels of γ in column 3 of the Table. Risk aversion is shown in column 4 (for $\alpha = 20$) and column 8 (for $\alpha = 70$). The first four rows of the table assume current wealth $w_0 = 100$ and the next four rows assume current wealth $w_0 = 200$. For all the examples we assume a 1-year horizon and an interest rate of 5%. The stock has a payoff with four states (120, 100, 80, 70) with equal probability. The option is a call option with a strike price of 100.

Given these data, we solve equations (3) for the optimal stock and option demands. For $\alpha = 20$, these are shown in columns 5 and 6 respectively. For

$\alpha = 70$, these are shown in columns 9 and 10 respectively. In part b) of the Table the results are shown for a different set of prices, $S_0 = 85$ and $a_0 = 3.70$. Then, in part c) they are shown for $S_0 = 86$ and $a_0 = 4.50$.

Observing the results, first note that the relative option demand, y/x, is unaffected either by wealth w_0 or by the subsistence parameter α. For example, given $C = 2.00$ in part b), $y/x = 0.23$ for all w_0 and α combinations. This illustrates the result of Rubinstein (1974). Looking at the column headed y, we observe that the option demand given $C = 0.25$ is never positive unless the demand given $C = 2.00$ is positive. Also, the option demand given $C = 2.00$ is only negative if the demand given $C = 0.25$ is negative. These results are consistent with Theorem 1.

Looking at the results in part a) or part b) it is tempting to conclude that the relative option demand y/x increases with C. However, the results in part c) of the Table show that this is not always the case. Given the prices $S_0 = 86$ and $a_0 = 4.50$, the short position in the option increases with C. However, the relative position y/x decreases (from -1.65 to -1.75). Note that here the payoff $xS + ya(S)$ is not monotonic. This case illustrates the need for the condition in Theorem 2.

7 Applications

In this section we consider two applications of the above results. First, stocks are call options on the assets of the firm. Also, corporate bonds are portfolios of the assets and options on the assets. It follows that we can use the three theorems to explain how cautiousness determines the demand for stocks v.s. corporate bonds. Second, in the HARA case, background risk increases cautiousness. Hence, we can use the three theorems to compare the derivative portfolios of agents with and without background risk.

7.1 Demand for Stocks v.s. Corporate Bonds

Consider a firm which has outstanding bonds and shares of stock. Assume the corporate bonds have no coupons and they all have the same maturity. Assume both the stock and the bonds are traded in the open market. Consider investors who invest their money in the firm's bonds and shares. Assume there is also a money market where they can borrow and lend money at the risk-free interest rate. Thus these investors can form portfolios of the money market instrument, the firm's bonds and shares. Assume the investors maximize their expected utility of their investments in such portfolios.

Let a_0 and B_0 be the initial total value of the stock and the bonds respectively. Let the value of the company at the maturity of the bond be S. Let the face value of the bonds be X. Then, as was first pointed out by Merton (1974), the total payoff of the stock at the maturity of the bonds is $(S-X)^+$, which implies that the shares are just call options on the firm value. Moreover, the payoff of the bonds at maturity is $S-(S-X)^+$. Denote investor i's initial wealth by w_{i0}. Assume he buys x_i fraction of the bonds and y_i fraction of the stock and invests the rest of his wealth in the money market. Then the value of his investment at maturity of the bonds is

$$w_i(S) = (w_{i0} - x_iB_0 - y_ia_0)(1+r) + x_i(S-(S-X)^+) + y_i(S-X)^+,$$

where r is the risk-free interest rate. Thus the investor decides his optimal portfolio strategy by solving problem (2), where $w_i(S)$ is defined above. We have the following result.

Proposition 1 *Assume investor i is more cautious than investor j and for both investor i and investor j there is a solution to problem (2). Then the following three statements are true.*

1. *Investor j holds a (strictly) positive position in the stock only if investor i does so, i.e., $y_j \geq (>)0$ only if $y_i \geq (>)0$, and investor i holds a (strictly) positive position in the corporate bonds only if investor j does so, i.e., $x_i \geq (>)0$ only if $x_j \geq (>)0$.*

2. *Investor j buys (strictly) more shares of the stock than the corporate bonds only if investor i does so, i.e., $y_j \geq (>) x_j$ only if $y_i \geq (>) x_i$.*

3. *If investor i holds a strictly positive position in the corporate bonds and in the stock, i.e., $x_i > 0$ and $y_i > 0$, then investor i holds a larger position in the stock per unit of his position in the bond than investor j, i.e., $\frac{y_i}{x_i} \geq \frac{y_j}{x_j}$, and if investor j holds a strictly negative position in the corporate bonds and in the stock, i.e., $x_j < 0$ and $y_j < 0$, then investor i holds a smaller short position in the stock per unit of his short position in the bond than investor j, i.e., $\frac{y_i}{x_i} \leq \frac{y_j}{x_j}$.*

Proof: To prove the first statement, we have $S - (S - X)^+ = X - (X - S)^+$ and $(S - X)^+ = S + (X - S)^+ - X$. Hence the terminal payoff of investor i's investment can be rewritten as

$$w_i(S) = A_i(x_i, y_i) + y_i S + (y_i - x_i)(X - S)^+, \tag{8}$$

where $A_i(x_i, y_i) = (w_{i0} - x_i B_0 - y_i a_0)(1+r) + (x_i - y_i)X$. In this case the original investment problem becomes a portfolio problem which involves a risk-free bond, a risky asset, and a put option on the risky asset. As investor i is more cautious than investor j, according to Theorem 3, investor j holds a (strictly) positive position in the risky asset only if investor i does so, i.e., $y_j \geq (>)0$ implies $y_i \geq (>)0$, and investor i holds a (strictly) larger position in the risky asset than in the call option only if investor j does so, i.e., $y_i - (y_i - x_i) = x_i \geq (>)0$ implies $y_j - (y_j - x_j) = x_j \geq (>)0$. Thus the first statement is proved.

In order to prove the second and third statements, we rewrite investor i's terminal payoff as

$$w_i(S) = (w_{i0} - x_i B_0 - y_i a_0)(1 + r) + x_i S + (y_i - x_i)(S - X)^+.$$

Thus the original investment problem is rewritten as a portfolio problem which involves a risk-free bond, a risky asset, and a call option on the risky asset. As investor i is more cautious than investor j, from Theorem 1, investor j holds a (strictly) positive position in the call option only if investor i does so, i.e.,

$y_j - x_j \geq (>)0$ implies $y_i - x_i \geq (>)0$. That is, investor j buys (strictly) more shares than bonds only if investor i buys (strictly) more shares than bonds. This proves the second statement.

To prove the third statement, notice that if $x_i > 0$, from the first statement, we must have $x_j > 0$. Thus if we also have $y_i > 0$, then $S + \frac{y_i - x_i}{x_i}(S - X)^+$ will be monotone. Now applying Theorem 2, we obtain $\frac{y_i - x_i}{x_i} \geq \frac{y_j - x_j}{x_j}$ which implies $\frac{y_i}{x_i} \geq \frac{y_j}{x_j}$. This proves the first half of the statement. The second half can be similarly proved. This completes the proof. Q.E.D.

Roughly speaking, Statement 1 of the above proposition tells us that a more cautious investor is more likely to buy stocks and less likely to buy corporate bonds. Statement 2 tells us that a more cautious investor is more likely to buy more shares of stocks than corporate bonds. Statement 3 tells us that a more cautious investor buys more shares of stocks relative to corporate bonds. The Proposition highlights the significance of cautiousness in explaining the demand for stocks and corporate bonds.

7.2 Impact of Background Risk on Demand for Options

We have shown that cautiousness measures an investor's motive to buy options. In this section we investigate the impact of background risk on an investor's cautiousness and hence on his motive to buy options. This topic has been discussed by FSS (1998), Gollier (2001), and HHK (2011).

Given a utility function, $u(x)$, when there is a background risk ϵ, as usual, we denote the derived utility function by $\hat{u}(x)$. For additive background risk ϵ, $\hat{u}(x) \equiv Eu(x + \epsilon)$, and for multiplicative background risk ϵ, $\hat{u}(x) \equiv Eu(x\epsilon)$.

Let $P(x)$ and $R(x)$ denote the absolute prudence and absolute risk aversion of the original utility function respectively. Let $\tilde{P}(x)$ and $\tilde{R}(x)$ denote the absolute prudence and absolute risk aversion of the derived utility function respectively. It has been shown that if for all x, $P(x) \geq kR(x)$, then for all x, $\tilde{P}(x) \geq k\tilde{R}(x)$, where $k > 0$ is a constant.[12] HHK derived a sufficient condition for the presence of additive background risk to increase an investor's cautiousness. They showed that if an investor has decreasing and convex cautiousness, then his cautiousness will be uniformly higher when exposed to additive background risk.

In this section we consider both additive and multiplicative background risks. We have the following result.

Lemma 2 (Carroll and Kimball (1996)) *Assume an investor's utility function $u(x)$ has positive third derivative and its cautiousness is higher than a constant. If the investor has an additive or multiplicative background risk, the cautiousness of the derived utility function will also be higher than the constant.*

This result was first presented by Carroll and Kimball (1996) in their seminal paper on the concavity of the consumption function. Its proof can be found in Carroll and Kimball (1996) or Gollier (2001).

Now combining the above lemma and the three main theorems in this paper we can obtain the following result.

Proposition 2 *Assume investors i and j have HARA class utility functions with identical constant cautiousness. If investor i has an additive or multiplicative background risk, then given any initial wealth, stock price, and derivative price such that there is a solution to problem (3) for both investors i and j, the following statements are true.*

1. *Investor j holds a (strictly) positive position in the derivative, i.e., $y_j \geq (>) 0$, only if investor i does so, i.e., $y_i \geq (>) 0$.*

2. *If $x_i > 0$, $x_j > 0$, and $S + \frac{y_i}{x_i} a(S)$ or $S + \frac{y_j}{x_j} a(S)$ is monotone, then investor i applies a larger weight to the option in his optimal risky portfolio, i.e., $\frac{y_i}{x_i} \geq \frac{y_j}{x_j}$, and if $x_i < 0$, $x_j < 0$, and $S + \frac{y_i}{x_i} a(S)$ or $S + \frac{y_j}{x_j} a(S)$ is monotone, then investor i applies a smaller weight to the derivative in his optimal risky portfolio, i.e., $\frac{y_i}{x_i} \leq \frac{y_j}{x_j}$.*

[12] See, for example, Proposition 23, page 115, Gollier (2001).

3. *Assume the derivative is a put option. If investor i buys (strictly) more shares of the stock than units of the put option, i.e., $x_i - y_i \geq (>)0$, then investor j buys (strictly) more shares of the stock than the option, i.e., $x_j - y_j \geq (>)0$.*

Proof: Note that a HARA utility function has constant cautiousness, say C. It follows from Lemma 2 that the cautiousness of the derived utility function is higher than C.[13] Hence investor i is more cautious than investor j. Now applying Theorems 1, 2, and 3, we immediately prove that the three statements are true. Q.E.D.

The above result explains the impact of background risk on portfolio strategies involving options. In particular, it shows that if the utility is HARA class then background risk strengthens an investor's motive to buy options.

FSS (1998) also studied the impact of background risk on an investor's demand for options. They focused on the case where background risk is additive. They showed that in an economy in which investors have identical constant positive cautiousness the investors without background risk will have globally concave optimal payoff functions, which they interpreted that background risk makes an investor more likely to buy options.

The difference between Proposition 2 and FSS's main result is worth noting although they give similar conclusions. FSS's model relies on the assumption that there is a complete market of contingent claims on the stock and the assumption that all investors are expected utility maximizers and have identical cautiousness, while Proposition 2 does not need these assumptions at all. Note Proposition 2 is even valid when many other investors are not rational utility

[13]Note in the proof of Proposition 2 the inequalities are strict for additive (multiplicative) background risk unless the utility function has constant absolute (relative) risk aversion, that is, it is exponential (power or logarithmic) utility. Hence if a utility function is HARA class, given an additive (multiplicative) background risk, the cautiousness of the derived utility function will be strictly higher unless the utility function is exponential (power or logarithmic) utility.

maximizers.

Lemma 2 can also be used to extend the main result in FSS (1998) to the case where investors have either additive or multiplicative background risk or both.

Moreover, combining Lemma 2 and Proposition 1, we can derive a result about the effect of background risk on the demand for stocks and corporate bonds. As in Proposition 2, consider two investors (i and j) who have HARA class utility functions with identical constant cautiousness. Suppose investor i has background risk while j does not. Then from Lemma 2, investor i is more cautious than investor j. Now applying Proposition 1, we conclude that the three statements of Proposition 1 are true. Roughly speaking, this result tells us that background risk has the following effects on the demand for stocks and corporate bonds: it makes an investor more likely to buy stocks and less likely to buy corporate bonds; it makes an investor more likely to buy more shares of stocks than corporate bonds; it makes an investor buy more shares of stocks relative to corporate bonds.

8 Conclusions

An individual investor's demand for derivatives depends upon a number of factors: the wealth of the investor, the prices of the stocks and the derivatives, and the utility function of the investor. In this paper we have considered the simple choice of investment in a stock, a risk-free bond and an option on the stock. We establish that it is the cautiousness of the investor's utility function that characterizes her demand for derivatives and in particular for put and call options. The cautiousness of a utility function is equivalent to the ratio of prudence to risk aversion. Comparing two investors, if i is more cautious than j, then i always buys options if j does so (Theorem 1). Also, if i is more cautious than j, then in most cases i buys more options per share of the stock than j (Theorem 2). Also, if i is more cautious than j, then if j over insures her portfolio using put options then so does i (Theorem 3).

REFERENCES

1. Arrow, Kenneth J. (1965), Aspects of a Theory of Risk Bearing, Yrjo Jahnsson Lectures, Helsinki. Reprinted in Essays in the theory of Risk Bearing (1971). Chicago: Markham Publishing Co.

2. Benninga S. and M. Blume (1985), On the Optimality of Portfolio Insurance, *Journal of Finance* 40, No. 5, 1341-1352.

3. Benninga, S. and J. Mayshar, (2000), Heterogeneity and Option Pricing, *Review of Derivatives Research* 4, 7-27.

4. Borch, K. (1962), Equilibrium in a Reinsurance Market, *Econometrica* 30, 424-444.

5. Brennan, M. J. and H. H. Cao (1996), Information, trade, and derivative securities, Review of Financial Studies 9, No.1, 163-208.

6. Brennan, M. J. and R. Solanki (1981), Optimal Portfolio Insurance, *Journal of Financial and Quantitative Analysis* 16, 279-300.

7. Carroll, C., and M. Kimball (1996), On the concavity of the Consumption Function, *Econometrica* 64, 981-992.

8. Carr P. and D. Madan, 2001, Optimal Positioning in Derivative Securities, *Quantitative Finance* 1, 19-37.

9. Eeckhoudt L. and H. Schlesinger (1994), A Precautionary Tale of Risk Aversion and Prudence. B. Munier and M. J. Machina (eds.), Models and Experiments in Risk and Rationality, 75-90, Kluwer Academic Publishers. Printed in the Netherlands.

10. Franke, G., R. C. Stapleton and M. G. Subrahmanyam (1998), Who Buys and Who Sells Options, *Journal of Economic Theory* 82, 89-109.

11. Franke, G., R. C. Stapleton, and M. G. Subrahmanyam (1999), When are Options Overpriced: The Black-Scholes Model and Alternative Characterizations of the Pricing Kernel, *European Finance Review* 3, 79-102.

12. Gollier, C. (2001), the Economics of Risk and Time, The MIT Press, London England.

13. Gollier, C. (2007), Whom Should We Believe? Aggregation of Heterogeneous Beliefs. *Journal of Risk and Uncertainty* 35, 107-127.

14. Gollier, C. (2008), Understanding Saving and Portfolio Choices with Predictable Changes in Assets Returns. *Journal of Mathematical Economics* 44, 445-458.

15. Gollier, C. and C. J. W. Pratt (1996), Risk Vulnerability and the tempering Effect of Background Risk, *Econometrica* 49, 1109-1123.

16. Huang C. and R. Litzenberger (1988), Foundations for Financial Economics. Prentice-Hall Canada, Incorporated.

17. Hara, C., J. Huang, and C. Kuzmics (2007), Representative Consumers Risk Aversion and Efficient Risk-Sharing. *Journal of Economic Theory* 137, 652-672.

18. Hara, C., J. Huang, and C. Kuzmics (2011), Effects of Background Risks on Cautiousness with an Application to a Portfolio Choice Problem. *Journal of Economic Theory* 146, 346-358.

19. Kihlstrome, Richard E., David Rome, and Steve Williams (1981): Risk Aversion with Random Initial Wealth, *Econometrica* 49, 911-920.

20. Kimball, Miles S. (1990), Precautionary Saving in the Small and in the Large, *Econometrica* 58, 53-73.

21. Kimball, Miles S. (1993), Standard Risk Aversion, *Econometrica* 61, 589-611.

22. Leland, H. E. (1980), Who Should Buy Portfolio Insurance? *Journal of Finance* 35, 581-594.

23. Mas-Colell, A. (1985), *The Theory of General Economic Equilibrium: A Differentiable Approach.* Econometric Society Monograph, Cambridge, Cambridge University Press.

24. Merton, R. C. (1974), On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance* 29, 449-470.

25. Pratt, J. W. (1964), Risk Aversion in the Small and in the Large, *Econometrica* 32, 122-136.

26. Pratt, J. W., and R. Zeckhauser (1987), Proper Risk Aversion, *Econometrica* 55, 143-154.

27. Rubinstein, M. E. (1976), An Aggregation Theorem for Securities Markets, *Journal of Financial Economics* 1, 225-244.

28. Wilson, R. (1968), The Theory of Syndicates, *Econometrica* 36, 119-132.

Tables

Table 1: a) Stock and Option Demand (84, 3.00)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	84	4	0.25	$\frac{4}{20+w}$	0.22	0.54	2.51	$\frac{4}{70+w}$	0.30	0.76	2.52
a_0	3.00	2	0.50	$\frac{2}{20+w}$	0.42	1.20	2.83	$\frac{2}{70+w}$	0.59	1.67	2.82
w_0	100	1	1.00	$\frac{1}{20+w}$	0.80	2.85	3.55	$\frac{1}{70+w}$	1.12	3.99	3.56
		0.5	2.00	$\frac{0.5}{20+w}$	1.36	7.48	5.50	$\frac{0.5}{70+w}$	1.91	10.48	5.50
S_0	84	4	0.25	$\frac{4}{20+w}$	0.40	1.00	2.52	$\frac{4}{70+w}$	0.49	1.22	2.52
a_0	3.00	2	0.50	$\frac{2}{20+w}$	0.78	2.20	2.83	$\frac{2}{70+w}$	0.95	2.69	2.84
w_0	200	1	1.00	$\frac{1}{20+w}$	1.47	5.24	3.55	$\frac{1}{70+w}$	1.80	6.37	3.55
		0.5	2.00	$\frac{0.5}{20+w}$	2.50	13.77	5.50	$\frac{0.5}{70+w}$	3.05	16.78	5.51

1. Table 1 a) shows the optimal stock and option demands given $(S_0, a_0) = (84, 3.00)$.
2. Investors have HARA utility with marginal utility $u'(w) = (w + \alpha)^{-\gamma}$ with $\alpha = 20,\ 70$

Table 1 b) Stock and Option Demand (85, 3.70)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	85	4	0.25	$\frac{4}{20+w}$	0.31	-0.06	-0.19	$\frac{4}{70+w}$	0.43	-0.08	-0.19
a_0	3.70	2	0.50	$\frac{2}{20+w}$	0.61	-0.08	-0.14	$\frac{2}{70+w}$	0.85	-0.11	-0.13
w_0	100	1	1.00	$\frac{1}{20+w}$	1.18	-0.03	-0.03	$\frac{1}{70+w}$	1.66	-0.05	-0.03
		0.5	2.00	$\frac{0.5}{20+w}$	2.21	0.50	0.23	$\frac{0.5}{70+w}$	3.08	0.71	0.23
S_0	85	4	0.25	$\frac{4}{20+w}$	0.56	-0.10	-0.19	$\frac{4}{70+w}$	0.69	-0.13	-0.19
a_0	3.70	2	0.50	$\frac{2}{20+w}$	1.12	-0.15	-0.14	$\frac{2}{70+w}$	1.36	-0.19	-0.14
w_0	200	1	1.00	$\frac{1}{20+w}$	2.18	-0.06	-0.03	$\frac{1}{70+w}$	2.65	-0.07	-0.03
		0.5	2.00	$\frac{0.5}{20+w}$	4.05	0.93	0.23	$\frac{0.5}{70+w}$	4.93	1.15	0.23

1. Table 1 b) shows the optimal stock and option demands given $(S_0, a_0) = (85, 3.70)$.
2. Investors have HARA utility with marginal utility $u'(w) = (w + \alpha)^{-\gamma}$ with $\alpha = 20,\ 70$

Table 1c) Stock and Option Demand (86, 4.50)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	86	4	0.25	$\frac{4}{20+w}$	0.47	-0.78	-1.65	$\frac{4}{70+w}$	0.66	-1.09	-1.65
a_0	4.50	2	0.50	$\frac{2}{20+w}$	0.95	-1.58	-1.67	$\frac{2}{70+w}$	1.33	-2.22	-1.67
w_0	100	1	1.00	$\frac{1}{20+w}$	1.92	-3.25	-1.70	$\frac{1}{70+w}$	2.69	-4.56	-1.70
		0.5	2.00	$\frac{0.5}{20+w}$	3.85	-6.74	-1.75	$\frac{0.5}{70+w}$	5.39	-9.44	-1.75
S_0	86	4	0.25	$\frac{4}{20+w}$	0.87	-1.43	-1.65	$\frac{4}{70+w}$	1.05	-1.74	-1.65
a_0	4.50	2	0.50	$\frac{2}{20+w}$	1.75	-2.91	-1.67	$\frac{2}{70+w}$	2.12	-3.54	-1.67
w_0	200	1	1.00	$\frac{1}{20+w}$	3.53	-5.99	-1.70	$\frac{1}{70+w}$	4.30	-7.29	-1.70
		0,5	2.00	$\frac{0.5}{20+w}$	7.08	-12.41	-1.75	$\frac{0.5}{70+w}$	8.62	-15.09	-1.75

1. Table 1 c) shows the optimal stock and option demands given $(S_0, a_0) = (86, 4.50)$.
2. Investors have HARA utility with marginal utility $u'(w) = (w + \alpha)^{-\gamma}$ with $\alpha = 20,\ 70$

Appendix A Proof of Lemma 1

We first prove that the two pricing kernels must intersect at least twice. By contradiction, suppose $\phi_i(S)$ crosses $\phi_j(S)$ only once at a from above.[14] We have

$$E(\phi_i(S) - \phi_j(S))S = E(\phi_i(S) - \phi_j(S))(S - a).$$

Suppose $\phi_i(S)$ and $\phi_j(S)$ are not identical, i.e., there exists a point b (in the support of the stock price distribution) such that $\phi_i(b) \neq \phi_j(b)$. As both $\phi_i(S)$ and $\phi_j(S)$ are continuous at $S = b$, there must exist a neighborhood of b with positive probability mass such that for all S in this set, $\phi_i(S) \neq \phi_j(S)$. This, together with the fact that $\phi_i(S) - \phi_j(S)$ is non-negative when $S < a$ and non-positive when $S > a$, implies that $E(\phi_i(S) - \phi_j(S))(S - a) < 0$. Thus we obtain $E(\phi_i(S) - \phi_j(S))S < 0$. This inequality contradicts the assumption that both pricing kernels price the stock correctly. This proves the first statement.

We now prove the second statement. Without loss of generality, assume $\phi_i(S)$ crosses $\phi_j(S)$ at b.[15] As both $\phi_i(S)$ and $\phi_j(S)$ are continuous, we must have $\phi_i(b) = \phi_j(b)$. From (6) we have

$$-\ln \frac{\phi_i(S)}{\phi_j(S)} = \int_b^S (\delta_i(x) - \delta_j(x))dx. \tag{9}$$

From the above equation if $\delta_i(S) - \delta_j(S)$ does not change sign at any point, then $\phi_i(S) - \phi_j(S)$ has opposite signs at the two sides of b. Thus $\phi_i(S) - \phi_j(S)$ can only change sign once at $S = b$. This contradicts the first statement. This proves the second statement.

We now prove the third statement. Assume $\delta_i(S)$ crosses $\delta_j(S)$ once at $S = b$. By contradiction, suppose $\phi_i(b) = \phi_j(b)$. From (9) it is clear that $\phi_i(S) - \phi_j(S)$ will have the same sign at the two sides of b. This implies $\phi_i(S)$ does not cross $\phi_j(S)$ which is impossible as both of their expectations are unity. This proves the third statement.

We now prove the fourth statement. As $\phi_i(S)$ must cross $\phi_j(S)$ and $\delta_i(S)$ must cross $\delta_j(S)$, without loss of generality, assume $\phi_i(S) - \phi_j(S)$ changes sign at b and $\delta_i(S) - \delta_j(S)$ changes sign from positive to negative at a. From the third statement it is clear that $a \neq b$.

[14] Note two pricing kernels must intersect at least once because otherwise their expectations cannot both be unity.

[15] See Footnote 11.

- Suppose $b < a$. Then by applying (9) we can see that *from left to right*, $\phi_i(S) - \phi_j(S)$ is negative at the left side of b, increases to zero at b, and keeps increasing to positive for $S \in [b, a)$. It starts to decrease at a; thus it can change sign at most one more time from positive to negative at the right side of a.
- Suppose $b > a$. Then again by applying (9) we can see that *from right to left*, $\phi_i(S) - \phi_j(S)$ is negative at the right side of b, increases to zero at b, and keeps increasing to positive for $S \in [a, b)$. It starts to decrease at a; thus it can change sign at most one more time from positive to negative at the left side of a.

Thus in both cases $\phi_i(S) - \phi_j(S)$ can change sign at most twice on the entire support. But according to the first statement, it must change sign at least twice. Thus it must change sign exactly twice. This completes the proof. Q.E.D.

Appendix B Proof of Theorem 2

To prove the theorem, we need the following lemma.

Lemma 3 *Assume* $1 + ya'(S) > (<)0$. *Let* $\hat{S} = S + ya(S) = h(S)$ *and* $\hat{a}(\hat{S}) \equiv a(h^{-1}(\hat{S}))$. *Then* $\hat{a}(\hat{S})$ *is a convex and piecewise twice differentiable function of* $\hat{S}$ *and strictly convex at* $\hat{S}^* \equiv S^* + ya(S^*)$.

Proof: As $\hat{S} = S + ya(S) = h(S)$ and $\hat{a}(\hat{S}) \equiv a(h^{-1}(\hat{S}))$, we have $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{a'(h^{-1}(\hat{S}))}{h'(h^{-1}(\hat{S}))}$. Simplifying, we obtain

$$\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{a'(S)}{1 + ya'(S)}. \tag{10}$$

As $a(S)$ is convex on the entire support and strictly convex at $S = S^*$, $a'(S)$ is increasing with S and strictly increasing at $S = S^*$. Because $S = h^{-1}(\hat{S})$ is strictly increasing with $\hat{S}$, this implies that $a'(S)$ is increasing with $\hat{S}$ and strictly increasing at $\hat{S} = \hat{S}^* \equiv h(S^*)$. But we have $\frac{d}{dx}\frac{x}{1+yx} = \frac{1}{(1+yx)^2} > 0$, i.e., for all x such that $1 + yx > (<)0$, $\frac{x}{1+yx}$ is a strictly increasing function of x. This implies that $\frac{a'(S)}{1+ya'(S)}$ is increasing with $\hat{S}$ and strictly increasing at $\hat{S} = \hat{S}^* \equiv h(S^*)$. From (10) it follows that $\frac{d\hat{a}(\hat{S})}{d\hat{S}}$ is increasing with $\hat{S}$ and strictly increasing at $\hat{S} = \hat{S}^*$. Thus we conclude that $\hat{a}(\hat{S})$ is a convex function of $\hat{S}$ and strictly convex at $\hat{S}^* \equiv h(S^*)$. Moreover, as $a(S)$ is piecewise twice differentiable, $\hat{a}(\hat{S})$ is also piecewise twice differentiable. Q.E.D.

With the help of the above lemma, we now prove the theorem. We first prove that the first statement implies the second statement.

When $x_i \neq 0$ and $x_j \neq 0$, let $\tilde{y}_i = y_i/x_i$ and $\tilde{y}_j = y_j/x_j$. Suppose $S + \tilde{y}_i a(S)$ is strictly monotone. Let $\hat{S} = h(S) \equiv S + \tilde{y}_i a(S)$. As $h(S)$ is strictly monotone, it follows that $S = h^{-1}(\hat{S})$. Let $\hat{a}(\hat{S}) \equiv a(h^{-1}(\hat{S}))$.

From Lemma 3, $\hat{a}(\hat{S})$ is a convex and piecewise twice differentiable function of $\hat{S}$ and strictly convex at $\hat{S}^* \equiv h(S^*)$. Thus the original investment problem with stock S and convex derivative $a(S)$ is transformed into a new investment problem with stock $\hat{S}$ and convex derivative $\hat{a}(\hat{S})$. From (1) in the original problem, investor i's terminal wealth is

$$w_i(S; x_i, \tilde{y}_i) = (w_{0i} - x_i(S_0 + \tilde{y}_i a_0))(1+r) + x_i(S + \tilde{y}_i a(S)),$$

and investor j's terminal wealth is

$$w_j(S; x_j, \tilde{y}_j) = (w_{0i} - x_j(S_0 + \tilde{y}_j a_0))(1+r) + x_j(S + \tilde{y}_j a(S)).$$

If we let $\hat{S}_0 \equiv S_0 + \tilde{y}_i a_0$ and $\hat{a}_0 \equiv a_0$, then in the transformed problem investor i's terminal wealth is $w_i(\hat{S}; x_i, 0) = (w_{0i} - x_i \hat{S}_0)(1+r) + x_i \hat{S}$, and investor j's terminal wealth is

$$w_j(\hat{S}; x_j, \tilde{y}_j - \tilde{y}_i) = (w_{0i} - x_j(\hat{S}_0 + (\tilde{y}_j - \tilde{y}_i)\hat{a}_0))(1+r) + x_j(\hat{S} + (\tilde{y}_j - \tilde{y}_i)\hat{a}(\hat{S})).$$

From the above two equations, we can clearly see that in the transformed problem investor i has x_i shares of the stock $\hat{S}$ and zero position in the convex derivative $\hat{a}(\hat{S})$ in his optimal portfolio while investor j's optimal positions in the stock $\hat{S}$ and the convex derivative $\hat{a}(\hat{S})$ are x_j and $x_j(\tilde{y}_j - \tilde{y}_i)$ respectively. Now assume investor i is more cautious than investor j. Applying Theorem 1 to the transformed problem, we immediately conclude that we must have $x_j(\tilde{y}_j - \tilde{y}_i) \leq 0$. If $x_j > (<)0$, this implies that $\tilde{y}_j - \tilde{y}_i \leq (\geq)0$, i.e., $\tilde{y}_i \geq (\leq)\tilde{y}_j$.

The proof for the case where $S + \tilde{y}_j a(S)$ is strictly monotone is similar. This proves that the first statement implies the second statement.

The proof of the converse is very similar to part of the proof for Theorem 1; thus it is omitted. Q.E.D.

Appendix C Proof of Theorem 3

To prove Theorem 3, we need the following two lemmas.

Lemma 4 *Assume investor i is more cautious than investor j. Then $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $S > K$ ($S < K$).*

Proof: Noting that for all $S < K$, $w_i''(S) = w_j''(S) = 0$, from (7) we have for all $S < K$, $\delta_t'(S) = -C_t(w_t(S))\delta_t^2(S)$, $t = i, j$. As for all w and v, $C_i(w) \geq C_j(v)$, from the above result we conclude that once $\delta_i(S) - \delta_j(S)$ becomes non-positive at $S = s_0 < K$, it will remain so for all $S \in (s_0, K)$. This implies that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $S < K$. Similarly, we conclude that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $S > K$. Q.E.D.

Lemma 5 *Assume that two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ both price the underlying stock correctly and cross each other twice. If one crossing happens at $s_1 < K$ and for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$, or one crossing happens at $s_2 > K$ and for almost every $S < K$, $\phi_i(S) \neq \phi_j(S)$, then the pricing kernel with a fatter right tail prices the convex derivative strictly higher.*

Proof: Without loss of generality, suppose $\phi_i(S)$ has a fatter right tail than $\phi_j(S)$. First assume one crossing happens at $s_1 < K$ and for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$. As the two pricing kernels cross each other twice, apart from s_1, they must also cross at s_2, where $s_2 \neq s_1$.

Now construct a portfolio of the money instrument and the stock such that its payoff is equal to the payoff of the derivative at s_1 and s_2. Denote the payoff of the portfolio by $L(S)$. We have

$$E(\phi_i(S) - \phi_j(S))a(S) = E(\phi_i(S) - \phi_j(S))(a(S) - L(S)). \tag{11}$$

First suppose $s_2 \leq K$. As both crossings happen at the same side of $S = K$, for all $S < K$, the portfolio has the same payoff as the derivative, i.e., $a(S) = L(S)$, while for all $S > K$, the portfolio has strictly lower payoff than the derivative, i.e., $a(S) > L(S)$. In the meantime, as $\phi_i(S)$ has a fatter right tail than $\phi_j(S)$, we must have for all $S > K$, $\phi_i(S) \geq \phi_j(S)$. But as for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$, we thus conclude that for almost every $S > K$, $\phi_i(S) > \phi_j(S)$.

It follows from (11) that

$$E(\phi_i(S) - \phi_j(S))a(S) = \int_{S>K} (\phi_i(S) - \phi_j(S))(a(S) - L(S))dP(S) > 0,$$

where $P(S)$ is the probability distribution function. This implies that $\phi_i(S)$ prices the derivative strictly higher than $\phi_j(S)$.

Now suppose $s_2 > K$. As $\phi_i(S)$ has a fatter right tail than $\phi_j(S)$, we must have $\phi_i(S) - \phi_j(S) \geq 0$, when $S < s_1$ or $S > s_2$; $\phi_i(S) - \phi_j(S) \leq 0$, when $s_1 < S < s_2$. But as for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$, we thus conclude that for almost every $S > s_2$, $\phi_i(S) > \phi_j(S)$. On the other hand, as $a(S)$ is convex while $L(S)$ is linear, we must have $a(S) - L(S) > 0$, when $S < s_1$ or $S > s_2$; $a(S) - L(S) < 0$, when $s_1 < S < s_2$. It follows from (11) that

$$E(\phi_i(S) - \phi_j(S))a(S) \geq \int_{S>s_2} (\phi_i(S) - \phi_j(S))(a(S) - L(S))dP(S) > 0,$$

i.e., $\phi_i(S)$ prices the derivative strictly higher.

When one crossing happens at $s_1 < K$ and for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$, the proof is similar; thus it is ommited for brevity. Q.E.D.

With the help of the above two lemmas we now prove Theorem 3. We only prove that Statement 1 implies Statements 2 and 3. The proof of the converse is very similar to part of the proof for Theorem 1; thus it is omitted for brevity.

We first prove that Statement 1 implies Statement 3. To prove that $x_j \geq 0$ implies $x_i \geq 0$, by contradiction, suppose that $x_j \geq 0$ while $x_i < 0$. This implies that for all $S > K$, $w_j'(S) \geq 0$ and $w_i'(S) < 0$. From (5) this further implies that for all $S > K$, $\delta_i(S) < 0$ and $\delta_j(S) \geq 0$. Thus we have for all $S > K$, $\delta_i(S) < \delta_j(S)$. But from Lemma 4, $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $S < K$. This implies that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above at some $s_0 \leq K$ in the entire support. Applying Statement 2 of Lemma 1, we conclude that $\delta_i(S)$ crosses $\delta_j(S)$ exactly once from above at some $s_0 \leq K$. Applying Statement 4 of Lemma 1, we conclude that $\phi_i(S)$ crosses $\phi_j(S)$ exactly twice and one crossing happens at some $s_1 < K$. Moreover, as for all $S > K$, $\delta_i(S) < \delta_j(S)$, we must have for almost every $S > K$, $\phi_i(S) \neq \phi_j(S)$. Now applying Lemma 5, we conclude that $\phi_i(S)$ and $\phi_j(S)$ cannot both price the option correctly, which causes a contradiction. Thus $x_j \geq 0$ must imply $x_i \geq 0$. Similarly, we can prove that $x_j > 0$ implies $x_i > 0$. This proves that Statement 1 implies Statement 3.

Similarly we can prove that Statement 1 implies Statement 2. Q.E.D.

Appendix D Proof of Theorem 1 (not for publication)

D.1 Two Lemmas

We first present two lemmas. These two lemmas will be used to prove that Statement 1 of Theorem 1 implies Statement 2 of Theorem 1. They will be also used in the proof of the converse.

As is explained in Section 3, y_i (y_j) denotes the units of the convex derivative in investor i's (j's) portfolio while $w_i(S)$ ($w_j(S)$) denotes investor i's (j's) terminal wealth. Also note that S^* is a point at which the derivative's payoff function $a(S)$ is strictly convex.

Lemma 6 *Assume either $y_i < 0$ and $y_j \geq 0$ or $y_i \leq 0$ and $y_j > 0$. If for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, then the following two statements are true.*

1. *Once $\delta_i(S) - \delta_j(S)$ becomes non-positive at $S = s_0$, it will remain so for all $S > s_0$.*
2. *Either there exists $\varepsilon > 0$ such that $\phi_i(S) \neq \phi_j(S)$ for all $S \in (S^* - \varepsilon, S^*) \cup (S^*, S^* + \varepsilon)$, or for all $S > S^*$, $\phi_i(S) > \phi_j(S)$.*

Proof:

Note $a(S)$ is globally convex in S and there exists at least one point, S^*, at which $a(S)$ is strictly convex. Thus investor j buys (sells) the derivative if and only if his optimal strategy is convex (concave) and the convexity (concavity) is strict convex for at least one point, S^*.

Suppose investor i holds a strictly negative position in the derivative but investor j holds a positive position in the derivative, then $w_i(S)$ is concave, and for at least one point, S^*, the concavity is strict; while $w_j(S)$ is convex.

Note $R_i(x)$ is differentiable and $w_i(S)$ is piecewise (twice) differentiable; thus in every differentiable interval, we have Equation (7), i.e., $\delta_i'(S) = (R_i(w_i(S))w_i'(S))' = -C_i(w_i(S))\delta_i^2(S) + R_i(w_i(S))w_i''(S)$. Hence if $(S, S+\tau)$ is contained in such an interval, then

$$\delta_i'(S+\tau) - \delta_i(S) = -\int_s^{s+\tau} C_i(w_i(S))\delta_i^2(S)dS + \int_S^{S+\tau} R_i(w_i(S))w_i''(S)dS, \quad (12)$$

$$\delta_j'(S+\tau)-\delta_j(S)=-\int_S^{S+\tau} C_j(w_j(S))\delta_j^2(S)dS+\int_S^{S+\tau} R_j(w_j(S))w_j''(S)dS \quad (13)$$

where $w_i''(S)\le 0$ while $w_j''(S)\ge 0$.

First consider any interval I in which the payoff of the derivative $a(S)$ is (twice) differentiable. Suppose at one point in this interval, say S, we have $\delta_i(S)=\delta_j(S)$. If S increases slightly by a small τ, since $C_i(w_i(s))\ge C_j(w_j(s))$ and $w_i''(S)\le 0$ while $w_j''(S)\ge 0$, then from (12) and (13), $\delta_i(S)$ decreases faster than $\delta_j(S)$, and we will have $\delta_i(S+\tau)\le\delta_j(S+\tau)$. We assert that the above inequality is true not only for small $\tau>0$ but also for all $\tau\in\{\tau|\tau>0,\ S+\tau\in I\}$. This is because after $\delta_i(S)$ becomes smaller than $\delta_j(S)$, if it somehow increases to the point such that they are close to each other again, then again $\delta_i(S)$ decreases faster than $\delta_j(S)$, and $\delta_i(S)$ stays smaller than $\delta_j(S)$ in the whole interval.

Now consider the points where $a'(S)$ has jumps. These jumps will cause jumps in $w_i(S)$ and $w_j(S)$ simultaneously. Since $\delta_i(S)=R_i(w_i(S))w_i'(S)$, where $R_i(w_i(S))$ is positive and globally continuous while $w_i'(S)$ is decreasing, when $\delta_i(S)$ jumps, it jumps down. For the opposite reason, when $\delta_j(S)$ jumps, it jumps up.

Hence combining the above two cases, we conclude that once $\delta_i(S)-\delta_j(S)$ becomes non-positive at $S=s_0$, it will remain so for all $S>s_0$. This proves the first statement.

We now prove the second statement. If $\phi_i(S^*)-\phi_j(S^*)\neq 0$, then as both $\phi_i(S)$ and $\phi_j(S)$ are continuous, there must exist $\varepsilon>0$ such that $\phi_i(S)-\phi_j(S)\neq 0$ for all $S\in(S^*-\varepsilon,S^*+\varepsilon)$. Thus we need only consider the case where $\phi_i(S^*)=\phi_j(S^*)$.

Suppose $a(S)$ is twice differentiable at S^*. Then $\phi_i(S^*)$ and $\phi_j(S^*)$ are twice differentiable at S^*. If $\delta_i(S^*)\neq\delta_j(S^*)$, then there must exist $\varepsilon>0$ such that $\phi_i(S)\neq\phi_j(S)$ for all $S\in(S^*-\varepsilon,S^*)\cup(S^*,S^*+\varepsilon)$. Thus we only need to consider the case where $\delta_i(S^*)=\delta_j(S^*)$. But in this case we have $\delta_i'(S^*)-\delta_j'(S^*)$ is equal to

$$\begin{aligned}&-[C_i(w_i(S^*))-C_j(w_j(S^*))]\delta_i^2(S^*)+R_i(w_i(S^*))w_i''(S^*)-R_j(w_j(S^*))w_j''(S^*)\\ &\le R_i(w_i(S^*))w_i''(S^*)-R_j(w_j(S^*))w_j''(S^*)<0\end{aligned}$$

The above strict inequality follows from the condition that $w_i(S)$ is strictly concave at S^* while $w_j(S)$ is convex at S^*. This strict inequality implies that $\delta_i(S)$ crosses $\delta_j(S)$ at S^*, but accroding to the third statement of Lemma 1, because $\phi_i(S^*)=\phi_j(S^*)$, this is ruled out.

Now suppose $a(S)$ is not twice differentiable at S^*. If $\delta_i(S^{*-})\neq\delta_j(S^{*-})$ and $\delta_i(S^{*+})\neq\delta_j(S^{*+})$, then as $\delta_i(S)$ and $\delta_j(S)$ are continuous in intervals $(S^*-\varepsilon,S^*)$

and $(S^*, S^* + \varepsilon)$ for some $\varepsilon > 0$, it is clear that there must exist $\varepsilon > 0$ such that $\phi_i(S) - \phi_j(S) \neq 0$ for all $S \in (S^* - \varepsilon, S^*) \cup (S^*, S^* + \varepsilon)$. Thus we need only consider the case where either $\delta_i(S^{*-}) = \delta_j(S^{*-})$ or $\delta_i(S^{*+}) = \delta_j(S^{*+})$.

Suppose $\delta_i(S^{*+}) = \delta_j(S^{*+})$. In this case as $\delta_i(S) = R_i(w_i(S))w_i'(S)$ and $\delta_j(S) = R_j(w_j(S))w_j'(S)$, where $w_i'(S)$ jumps down at $S = S^*$ while $w_j'(S)$ strictly jumps up at $S = S^*$, we must have $\delta_i(S^{*-}) > \delta_j(S^{*-})$. As an immediate consequence of the first statement, we can conclude that $\delta_i(S)$ crosses $\delta_j(S)$ once at $S = S^*$. But according to the third statement of Lemma 1, because $\phi_i(S^*) = \phi_j(S^*)$, this is ruled out.

Now suppose $\delta_i(S^{*-}) = \delta_j(S^{*-})$.

Similar to the above case, we must have $\delta_i(S^{*+}) < \delta_j(S^{*+})$. Thus from the first statement for all $S > S^*$, $\delta_i(S) \leq \delta_j(S)$. Moreover, as $\delta_i(S)$ and $\delta_j(S)$ are differentiable at the points near S^* there must exist $\epsilon > 0$ such that for all $S \in (S^*, S^* + \varepsilon)$, $\delta_i(S^{*+}) < \delta_j(S^{*+})$. Applying Equation (9) we find that for all $S > S^*$, $\phi_i(S) > \phi_j(S)$. This completes the proof. Q.E.D.

Lemma 7 *Assume two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ both price the underlying stock correctly. Assume either there exists $\varepsilon > 0$ such that $\phi_i(S) \neq \phi_j(S)$ for all $S \in (S^* - \varepsilon, S^*) \cup (S^*, S^* + \varepsilon)$, or for all $S > S^*$, $\phi_i(S) > \phi_j(S)$. If the two pricing kernels intersect each other twice, then the pricing kernel with a fatter right tail prices the derivative whose payoff is strictly convex at S^* strictly higher.*

Proof:

Without loss of generality, suppose $\phi_i(S)$ has a fatter right tail. As the two pricing kernels intersect exactly twice, there exist S_1 and S_2, where $S_1 < S_2$, such that $\phi_i(S) - \phi_j(S) \geq 0$, when $S < S_1$, or $S > S_2$; $\phi_i(S) - \phi_j(S) \leq 0$, when $S_1 < S < S_2$.

Now construct a portfolio of the money instrument and the stock such that its payoff is equal to the payoff of the convex derivative at S_1 and S_2. Denote the payoff of the portfolio by $L(S)$. As $L(S)$ is linear and both $\phi_i(S)$ and $\phi_j(S)$ price the underlying stock correctly, we must have

$$E(\phi_i(S) - \phi_j(S))L(S) = 0.$$

It follows that

$$E(\phi_i(S) - \phi_j(S))a(S) = E(\phi_i(S) - \phi_j(S))(a(S) - L(S)).$$

If we notice that $a(S)$ is convex while $L(S)$ is linear, then it can be easily verified that $\phi_i(S) - \phi_j(S)$ and $a(S) - L(S)$ can never have opposite signs.

Moreover, as is assumed, either there exists $\varepsilon > 0$ such that $\phi_i(S) \neq \phi_j(S)$ for all $S \in (S^* - \varepsilon, S^*) \cup (S^*, S^* + \varepsilon)$, or for all $S > S^*$, $\phi_i(S) > \phi_j(S)$. Consider the first case. from the strict concavity of $w_i(S)$ at $S = S^*$, there exists either interval $(S^* - \varepsilon, S^*)$ or interval $(S^*, S^* + \varepsilon)$, where $\varepsilon > 0$, such that $a(S) - L(S) \neq 0$ on the entire interval. Thus there must exist $\varepsilon > 0$ such that we have $a(S) - L(S) \neq 0$ and $\phi_i(S) - \phi_j(S) \neq 0$ for all $S \in (S^* - \varepsilon, S^*)$ or for all $S \in (S^*, S^* + \varepsilon)$. This, together with the fact that $\phi_i(S) - \phi_j(S)$ and $a(S) - L(S)$ can never have opposite signs, implies that the expectation $E[(\phi_i(S) - \phi_j(S))a(S)]$ is strictly positive, i.e., $\phi_i(S)$ prices the derivative strictly higher.

Now consider the second case. As crossings between $\phi_i(S)$ and $\phi_j(S)$ only happen to the left of S^*, the strict convexity of $a(S)$ at S^* implies that there must exist $\varepsilon > 0$ such that $a(S) - L(S) \neq 0$ for all $S \in (S^*, S^* + \varepsilon)$. But as it is assumed that for all $S > S^*$, $\phi_i(S) > \phi_j(S)$, this again implies that the expectation $E[(\phi_i(S) - \phi_j(S))a(S)]$ is strictly positive, i.e., $\phi_i(S)$ prices the derivative strictly higher. Q.E.D.

D.2 Statement 1 $\Rightarrow$ Statement 2

With the help of the above two lemmas we can now prove that Statement 1 of Theorem 1 implies Statement 2 of Theorem 1:

By contradiction, suppose investor i holds a negative position in the derivative ($y_i \leq 0$) while investor j holds a strictly positive position in the derivative ($y_j > 0$) or investor i holds a strictly negative position in the derivative ($y_i < 0$) while investor j holds a positive position in the derivative ($y_j \geq 0$).

From the first statement of Lemma 6 we know that $\delta_i(S) - \delta_j(S)$ changes sign at most once. But according to the second statement of Lemma 1, it must change sign at least once. Thus $\delta_i(S) - \delta_j(S)$ changes sign exactly once. From the fourth statement of Lemma 1, we conclude that $\phi_i(S) - \phi_j(S)$ changes sign exactly twice. Moreover, from Lemma 6, either there exists $\varepsilon > 0$ such that $\phi_i(S) \neq \phi_j(S)$ for all $S \in (S^* - \varepsilon, S^*) \cup (S^*, S^* + \varepsilon)$, or, for all $S > S^*$, $\phi_i(S) > \phi_j(S)$. Now applying Lemma 7, we find that the two pricing kernels cannot both correctly price the derivative, whose payoff is strictly convex at S^*. This causes a contradiction. Q.E.D.

D.3 Statement 2 $\Rightarrow$ Statement 1

Before we start to prove that Statement 2 of Theorem 1 implies Statement 1 of Theorem 1, consider the following explanation. Recall that we have assumed that the expected-utility-maximizing investors in the market are strictly risk-averse. In the rare case where the current prices of the stock and the derivative are equal to the risk neutral prices, a strictly risk averse investor will optimally hold zero investment in both the stock and the derivative. Thus if we use S_r and a_r to denote the risk neutral prices of the stock and the derivative respectively, when $(S_0, a_0) = (S_r, a_r)$, a solution to (3) is $(x_i, y_i) = (0, 0)$. We now show that for those (S_0, a_0) which are near (S_r, a_r), solutions to (4) exist too. We have the following lemma.

Lemma 8 *The following two statements are true.*

1. *There exists a neighborhood of $(0,0)$, B, such that for any $(x_i, y_i) \in B$, there exists (S_0, a_0) such that (x_i, y_i) is the solution to (4).*
2. *There exists a neighborhood of (S_r, a_r), A, such that for any $(S_0, a_0) \in A$, a solution to (4) exists.*

Proof:

We first prove Statement 1. Since the support of the stock price distribution is a bounded subinterval of $[0, +\infty)$, without loss generality denote it by $[a, b]$. As the support is bounded the prices of the stock and the derivative under the first stochastic dominance rule must be bounded. Let $\underline{S}$ and $\bar{S}$ be the lower and upper bounds of the stock price; let $\underline{a}$ and $\bar{a}$ be the lower and upper bounds of the derivative price.[16] Consider the problem in which given a pair of (x_i, y_i), we want to solve (4) for (S_0, a_0). Define the function

$$g(S_0, a_0) = \frac{1}{1+r}(E[\phi_i(w_i(S))S], E[\phi(w_i(S))a(S)]).$$

For any pair of (x_i, y_i) which is close enough to $(0,0)$, this function is well defined on $[\underline{S}, \bar{S}] \times [\underline{a}, \bar{a}]$. As utility functions are three times differentiable, $g(.)$ is obviously a sequentially continuous function, thus a continuous function of a non-empty, closed,

[16] It is straightforward that $\underline{S} = \frac{a}{1+r}$, $\overline{S} = \frac{b}{1+r}$, $\underline{a} = \min_{x \in [a,b]} \frac{a(x)}{1+r}$, and $\overline{a} = \max_{x \in [a,b]} \frac{a(x)}{1+r}$.

bounded, convex set $[\underline{S}, \bar{S}] \times [\underline{a}, \bar{a}]$ into itself. According to the well-known Brouwer's Fixed Point Theorem, there is always a fixed point. Thus a solution of (S_0, a_0) to (4) always exists. This proves the first statement.

We now prove the second statement. Define a function $f(.)$ on $[0, +\infty) \times [0, +\infty)$ as follows. For a pair of stock price and derivative price (S_0, a_0), if there is a solution (x_i, y_i) to (4), then $f(S_0, a_0) = (x_i, y_i)$. Note as is well-known, because of the strict concavity of utility function $u_i(w)$, the solution (x_i, y_i) is unique; thus the function is well defined. As utility functions are three times differentiable, $f(.)$ is obviously sequentially continuous. But in a metric space sequential continuity and continuity are equivalent; thus $f(.)$ is continuous.

From the first statement we conclude that there is a neighborhood of $(0, 0)$, B, such that B is a set of images under function $f(.)$. Since $f(.)$ is continuous and B is open, the preimage of B is also open. Thus as $f(S_r, a_r) = (0, 0)$ there must exist a neighborhood of (S_r, a_r), A, such that for any $(S_0, a_0) \in A$, a solution to (4) exists. Q.E.D.

With the help of the above lemma we can now start to prove that Statement 2 of Theorem 1 implies Statement 1 of Theorem 1. Note that if there does not exist a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$, then there must exist some w_0 and v_0 such that $C_i(w_0) < C_j(v_0)$. As all utility functions are assumed to be three times continuously differentiable, cautiousness is continuous; Thus there must be a neighborhood of w_0, A, a neighborhood of v_0, B, and a constant α, such that for all $w \in A$ and all $v \in B$, $C_i(w) < \alpha < C_j(v)$. If we can somehow make sure that investor i's terminal wealth is contained in A while investor j's terminal wealth is contained in B, then using an argument very similar to the proof that Statement 1 of Theorem 1 implies Statement 2 of Theorem 1, we can show a situation where it happens that investor j optimally holds a long position in the derivative while i does not. This is the idea we use to prove that Statement 2 of Theorem 1 implies Statement 1 of Theorem 1.

We need only show that if there does not exist a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$ then there is a set of w_{i0}, w_{j0}, S_0, and a_0 such that investor j optimally holds a long position in the derivative while i does not.

Applying the first statement of Lemma 8, we conclude that there is a series: $\{(x_i^n, 0)|n = 1, 2, ...\}$, where x_i^n is strictly decreasing in n, $\lim_{n\to\infty} x_i^n = 0$, and for all n, $(x_i^n, 0)$ is the solution to (4) corresponding to $(S_0, a_0) = (S_{0n}, a_{0n})$. Obviously we have

$$\lim_{n\to\infty} S_{0n} = S_r \text{ and } \lim_{n\to\infty} a_{0n} = a_r.$$

According to the second statement of Lemma 8, there exists a neighborhood of (S_r, a_r), A, such that for any $(S_0, a_0) \in A$, the solution to problem (3) exists. Without loss of generality assume for all n, $(S_{0n}, a_{0n}) \in A$. This implies that given the series $\{(S_{0n}, a_{0n})|n = 1, 2, ...\}$, there exist a series of solutions $\{(x_{jn}, y_{jn})|n = 1, 2, ...\}$ to problem (3) for investor j. Since $\lim_{n\to\infty}(S_{0n}, a_{0n}) = (S_r, a_r)$ from the continuity of the solutions we have

$$\lim_{n\to\infty}(x_{jn}, y_{jn}) = (0, 0).$$

As is pointed out in the paragraph preceding this proof, since there does not exist a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$, from the continuity of $C_i(w)$ and $C_j(v)$, there must be w_0, v_0, a neighborhood of w_0, A, a neighborhood of v_0, B, and a constant α, such that for all $w \in A$ and all $v \in B$, $C_i(w) < \alpha < C_j(v)$. Let $w_{i0} = w_0/(1 + r)$ and $w_{j0} = v_0/(1 + r)$.[17] Use $w_{in}(S)$ to denote investor i's terminal wealth corresponding to trading strategy $(x_i, y_i) = (x_{in}, y_{in})$, which is defined in Equation (1). Then since the support of the stock price distribution, $[a, b]$, is bounded, there must exist $N > 0$ such that for all $n > N$, we have that for all $S \in [a, b]$, $w_{in}(S) \in A$ and $w_{jn}(S) \in B$.

This implies that for all $S \in [a, b]$, $C_i(w_{in}(S)) < \alpha < C_j(w_{jn}(S))$. Now we assert that for all $n > N$ we must have $y_{jn} > 0$. By contradiction, suppose for some $n > N$, $y_{jn} \leq 0$.

Applying the first statement of Lemma 6, we know that $\delta_i(S) - \delta_j(S)$ changes sign at most once. But according to the second statement of Lemma 1, it must change sign at least once. Thus $\delta_i(S) - \delta_j(S)$ changes sign exactly once. From the third statement of Lemma 1, we conclude that $\phi_i(S) - \phi_j(S)$ changes sign twice. Moreover, from the second statement of Lemma 6, there exists a neighborhood of S^*, A, such that $\phi_i(S) - \phi_j(S) \neq 0$ for all $S \in A \setminus \{S^*\}$. Now applying Lemma 7, we find that the two pricing kernels cannot both price the derivative correctly. This causes a contradiction.

[17] To disallow negative wealth, we need only restrict the natural domains of the utility functions to $[0, +\infty)$. This restriction does not have any effect on the proof.

Hence for $n > N$ we must have $y_{jn} > 0$. Thus we have a situation where investor j holds a (strictly) positive position in the derivative, but investor i does not do so. This completes the proof. Q.E.D.