Risk-Taking-Neutral Background Risks

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Abstract

This paper examines how decision making under uncertainty is affected by the presence of a background risk. By background risk, we refer to a risk for which there is no market for trading or hedging. In particular, we construct a class of background risks that we label as risk-taking-neutral (RTN). These background risks have the property that they will not alter the choice decisions made with respect to another risk. As such, these RTN background risks can provide a benchmark. In many situations, a background risk that is faced by an investor can be compared to one from the RTN class in order to predict qualitative changes in the investor's choice decision. In particular, we illustrate our benchmarking with three examples with regards to portfolio choice: (1) effects of a flat-rate income tax, (2) effects of an independent non–positive-mean background risk, and (3) a theorem about dynamic investing due to Mossin (1968a).

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1 Introduction

Consider an individual who must make an economic decision in the face of risk. For example, the individual might be an investor deciding on how to allocate wealth between a risky and a riskfree asset. Alternatively, the individual might be deciding on how to insure or how to otherwise hedge a risky asset. Typically, such a decision is modeled in isolation, where there is only the one source of risk. For the sake of clarity, we will refer to this risk as the "primary risk." However, more realistically, there are other risks that are also faced by the individual. One type of such risk is often referred to as a "background risk," meaning that there is no market for trading directly on this second risk. One question that has been given considerable attention in the literature is "how does the presence of this background risk affect behavior towards the primary risk?" Obviously, many types of dependence between the risks might be indirectly treated via trading on the primary risk. For example, contracts on the primary risk might partially mitigate effects of the background risk via "cross hedging" techniques. However, even in cases where such techniques are not possible, such as when the two risks are independent, it is now well known that a background risk can still affect decisions about the treatment of the primary risk.[1]

[1]This line of research began with Kihlstrom et al. (1981), Ross (1981) and Nachman (1982). These papers all considered one individual who was more risk averse than another with resepect to the primary risk and examined whether the individual remained more risk averse (with respect to the primary risk) in the presence of such a background risk. Doherty and Schlesinger (1983) showed how such a background risk might affect an individual's decision towards the primary risk. Good summaries of how such background

For example, one might hypothesize that an independent background risk with a non-positive mean will lead to less risk taking with respect to the primary risk. However, as shown by Gollier and Pratt (1996), risk aversion alone is not sufficient to guarantee such behavior. Preferences that do guarantee such behavior are labeled as "risk vulnerable" by Gollier and Pratt (1996), who also derive the tedious necessary and sufficient conditions that lead to such behavior. Luckily, relevant sufficient conditions are much easier to satisfy. A few papers have looked at cases where the two risks are not independent, but these cases are usually very restrictive.[2] As another illustration, consider investment decisions modelled in a world without taxes. How would the inclusion of an income tax affect investment strategy?

In this paper, we construct a class of background risks for an individual with the property that they will not affect an individual's decision making on the primary risk. In particular, we construct this class of background risks, which we label as "risk-taking-neutral background risk," for expected-utility preferences with so-called "hyperbolic absolute risk aversion" (HARA). The closure of the HARA class of utility includes most all of the commonly used utility functions, including constant absolute risk aversion (CARA), constant relative risk aversion (CRRA) and quadratic utility. The importance of this class within finance includes it's equivalence to the set of utility functions allowing for two-fund separation in portfolio choice, as shown by Cass and Stiglitz (1970). Moreover, the HARA class of utility allows for dynamic portfolio choices to be made somewhat myopically, "as if" future period returns were all risk-free. See Mossin (1968a) and Gollier (2001). Indeed, we will prove Mossin's (1968a) main result later in the paper, as an example of risk-taking-neutral background risk.

The terminology "background *risk*" is actually a bit more narrow than our construction, since we also allow for deterministic transformations of wealth. But since these "deterministic transformations" are themselves risky, since they vary with wealth, we still refer to them as a background risk. Although this class of RTN background risks is fairly simple to construct, such background risks need not resemble the types of background risks that might

risks embed into economic and financial decisions can be found in Campbell and Viceira (2002) and Gollier (2001).

[2] See, for example, Dana and Scarsini (2007).

occur naturally within an economy. However, these RTN background risks can serve as a benchmark. That is, in many situations, a canonical type of background risk can be compared to a member of the RTN class in such a way as to predict qualitative changes in risk-taking behavior towards the primary risk. Indeed, it might be more difficult to compare this canonical background risk directly with no background risk. The comparison with a suitably chosen RTN background risk might make economic analysis much simpler.

We begin the next section with a description of our class of RTN background risks within the class of HARA utility. We next show how several examples of background risks can be analyzed via comparison to our RTN class of background risks. In particular, we examine portfolio choice in a few different settings. We first examine the effect of an income tax on optimal portfolio choice and show how it can be easily modelled using our RTN background risk. The result is also robust to having a tax rate that is random in the sense of being unknown at the time investment decisions are made. We next consider an independent, non-positive-mean background risk, as in Gollier and Pratt (1996). By choosing an appropriate RTN background risk, which by definition does not affect investment decisions, we can easily show the logic behind the Gollier and Pratt result. Finally, we show how a famous result about dynamic portfolio choice, due to Mossin (1968a), follows in straightforward manner using RTN background risk.

2 Risk-Taking-Neutral Background Risk

Consider a risk-averse individual with random wealth $\widetilde{x}$ (the "primary risk") who maximizes her expected utility of terminal wealth . We assume that utility belongs to the so-called hyperbolic-absolute-risk-aversion (HARA) class of utility, where we can express utility as

$$u(x) = \frac{1}{B-1}(A + Bx)^{1-\frac{1}{B}}, \qquad (1)$$

where $B \neq 0$, $B \neq 1$ and $A + Bx > 0$. Risk tolerance for this class of utility is linear:

$$\frac{-u'(x)}{u''(x)} = A + Bx. \tag{2}$$

Note that this class of utility includes constant relative risk aversion ($A = 0$) and quadratic utility ($B = -1$). It also is straightforward to show, using L'Hospital's rule, that such utility approaches constant-absolute-risk-aversion (CARA) utility as $B \to 0$, $u(x) = -e^{-kx}$, with $k = \frac{1}{A}$. Also, utility approaches log utility, $u(x) = \ln x$, if $A = 0$ and $B \to 1$.

We assume that the random wealth $\widetilde{x}$ is endogenous in that the individual can engage in market activity affecting the distribution of terminal wealth. For the sake of concreteness, we will use the case where the individual allocates her wealth between a risky asset and a riskfree asset. From standard portfolio-choice analysis, we know that a risk-averse individual always invests a positive amount in the risky asset if and only if its expected payoff per dollar invested is higher than the payoff on the riskfree asset. Moreover, a more risk-averse individual would always invest less in the risky asset, *ceteris paribus*.

We now suppose that the investor faces a second risk $\widetilde{z}$, the so-called "background risk" for which there is no market available for trading and/or hedging. Final wealth is denoted as $\widetilde{x}+\widetilde{z}$. The question addressed in this paper is whether or not we can predict that the individual take less [or more] risk with respect to the primary risk $\widetilde{x}$ in the market in the presence of background risk $\widetilde{z}$.

To facilitate answering such a question, we define the class of risk-taking-neutral (RTN) background risks to be any risk of the following form

$$\widetilde{z}(x) = (k + \widetilde{\varepsilon})(\frac{A}{B} + x), \tag{3}$$

where $k \in \mathbb{R}$ and the random variable $\widetilde{\varepsilon}$ is statistically independent of the random variable $\widetilde{x}$, with $E\widetilde{\varepsilon} = 0$, where E denotes the expectation operator. We further require that $1 + k + \varepsilon > 0$ for all values of ε. Note that we allow for the possibility that $\widetilde{\varepsilon}$ is degenerate, with variance zero. Although $\widetilde{\varepsilon}$ and $\widetilde{x}$ are statistically independent by assumption, $\widetilde{\varepsilon}$ and $\widetilde{z}$ are statistically dependent by construction.

It is important to note that our construction of $\widetilde{z}$ is parametric, based on the parameters in the decision maker's utility function u. It is also important to note that only the ratio $\frac{A}{B}$ matters in our construction. This is important since our definition of HARA in (1) is unique only up to an affine transformation of u.

To see the effect of $\widetilde{z}$ on decisions made about the primary risk $\widetilde{x}$, we consider the so-called derived utility function[3] for an arbitrary background risk $\widetilde{z}$:

$$v(x) \equiv Eu(x + \widetilde{z}). \tag{4}$$

For $\widetilde{z}$ belonging to the RTN class of background risk, we obtain

$$v(x) = E\left[\frac{1}{B-1}((A + Bx)(1 + k + \widetilde{\varepsilon}))^{1-\frac{1}{B}}\right] = E(1 + k + \widetilde{\varepsilon}))^{1-\frac{1}{B}} \cdot u(x). \tag{5}$$

Since $\widetilde{\varepsilon}$ is statistically independent of $\widetilde{x}$, decisions made about the primary risk $\widetilde{x}$ in the presence of background risk are identical to decisions on $\widetilde{x}$ without background risk, but using the derived utility function v in place of u. That is

$$Eu(\widetilde{x} + \widetilde{z}) = Ev(\widetilde{x}) = E(1 + k + \widetilde{\varepsilon})^{1-\frac{1}{B}} \cdot Eu(\widetilde{x}), \tag{6}$$

where $E(1 + k + \widetilde{\varepsilon})^{1-\frac{1}{B}}$ is a positive constant.

In any decision problem about $\widetilde{x}$, we can interpret the first-order conditions as setting marginal benefits equal to marginal costs, for changes in the decision variable, where benefits and costs are given in terms of marginal utility. Since both marginal benefits and marginal costs are simply scaled by the common multiplicative factor $E(1 + k + \widetilde{\varepsilon})^{1-\frac{1}{B}}$ in the presence of $\widetilde{z}$, the optimal decision remains unchanged. It is important to note that we are not claiming that preferences are unaffected by $\widetilde{z}$. If $E(1 + k + \widetilde{\varepsilon})^{1-\frac{1}{B}} < 1$, then the individual is worse off when background risk $\widetilde{z}$ is present. If the inequality is reversed, the individual is better off. However, decisions made about $\widetilde{x}$ are not affected whether $\widetilde{z}$ is present or not, hence our terminology "risk-taking-neutral background risk."

[3]See Kihlstrom et al. (1981), who refer to this function as the indirect utility function, and Nachman (1982). Note also that this background risk contains a mixture of an additive and multiplicative background risk, as described by Franke et al. (2011).

The types of background risks a decision might typically face are not likely to belong to the class of risks given by (3) above, although we will show later that they sometimes might be. The main purpose in defining our class of RTN background risks is to provide a benchmark. By appropriately choosing a member of the RTN class, one might be able to make a simple comparison with an actual background risk to qualitatively determine how decisions will change. In other words, comparing a decision under an actual background risk might be easier against an RTN background risk than against no background risk. And the latter two yield the same optimal decisions. The rest of the paper provides a few examples of how RTN background risks can be used in different settings.

3 Portfolio Choice and Taxes

In this section, we show how our RTN background risks can be used to re-examine an age-old problem in public finance: the question of how taxes affect investment in risky assets. Although there has been much research on the effects of taxes on portfolio choice over the years, most of it has focussed on the effects of differing tax rates for different asset classes. But some early research has focused on the effects of a simple flat-rate income tax on portfolio choice. Here we consider a theoretical model where there is a fixed tax rate on one's income. We show why, under fairly broad circumstances, an income tax will increase the investment in the risky asset. We then extend our results to the case where the tax rate is uncertain at the time portfolio choices are made.

Domar and Musgrave (1944) were the first to consider the problem. They argued, against conventional wisdom at the time, that income taxes were most likely to increase investment in risky assets, rather than decrease it. The basic model was formalized in an expected-utility setting by Mossin (1968b) and by Stiglitz (1969).[4]

For the sake of clarity, we consider a particular choice problem; namely the

[4] See Sandmo (2010) for an excellent summary, discussion and extension of these early results.

choice of allocating wealth between a risky and a riskfree asset. To this end, let w denote the initial wealth of the individual who must decide on the amount of wealth a to invest in the risky asset, with the remaining wealth $w-a$ invested in the riskfree asset. The gross return on the riskfree asset is denoted as $R_f \geq 1$, whereas the risky asset's gross return is denoted by the random variable $\widetilde{R}$, where we assume that $\widetilde{R} \geq 0$ and $E\widetilde{R} > R_f$. This last assumption guarantees that the optimal investment in the risky asset $a*$ will be strictly positive. Note that, unlike in Mossin (1968a), we do not restrict $R_f = 1$.

To implement our analysis, consider the following RTN background "risk": $z(x) = -t(\frac{A}{B} + x)$, where $0 < t < 1$. In other words, we let $\widetilde{\varepsilon}$ be identically zero and define $k \equiv -t$. Since $0 < t < 1$, our constraint that $(1+k+\varepsilon)^{1-\frac{1}{B}} > 0$ is trivially satisfied. We assume for now that $B > 0$. Obviously, in this example z is not random for a fixed level of wealth x. But since $z(\widetilde{x})$ varies with $\widetilde{x}$, we will still refer to z as a RTN "background risk," since z satisfies our definition (3). In this case we obtain

$$Eu(\widetilde{x} + \widetilde{z}) = (1-t))^{1-\frac{1}{B}} \cdot Eu(\widetilde{x}). \tag{7}$$

Note that we can write total random wealth in this case as

$$\widetilde{x} + \widetilde{z} = \widetilde{x}(1-t) - t\frac{A}{B}. \tag{8}$$

Recall that, by construction, as is clear from (7), the optimal choice of investment in the risky asset $a*$ is the same both with and without the background risk $\widetilde{z}$.

Consider first the case of a wealth tax, with tax rate t. That is, the individual must pay a fixed percent of her final wealth as a tax. Using (8) we can write after-tax wealth as follows:

$$\widetilde{x}(1-t) = (\widetilde{x} + \widetilde{z}) + t\frac{A}{B}. \tag{9}$$

First consider the case where $A = 0$, so that we have CRRA. In this case, we see that $\widetilde{x}(1-t) = (\widetilde{x} + \widetilde{z})$, so that there is no effect of a wealth tax on the optimal optimal portfolio choice. That is, the optimal investment in the risky asset $a*$ is unchanged by the wealth tax. This effect is as expected under CRRA since wealth is reduced proportionally at every final wealth level.

Now suppose that $A > 0$. We can proceed in two steps: (1) First note that changing wealth from $\widetilde{x}$ to $\widetilde{x} + \widetilde{z}$ will not affect the optimal choice of risky investment $a*$, since $\widetilde{z}$ is RTN. (2) Second, if we add the positive constant $t\frac{A}{B}$ to every wealth level, it will increase the optimal risky investment $a*$, since $B > 0$ implies that absolute risk aversion is decreasing (DARA). Hence, the total effect of the wealth tax is to increase investment in the risky asset. This two-step procedure makes full use of our RTN background risk set-up, since the second step does not require us to compare a situation with the tax to one with no tax; rather we compare after-tax wealth to the RTN alternative $\widetilde{x} + \widetilde{z}$.

At first thought, it might seem like this result is simply due to DARA. However the result does not hold in general for any DARA utility, unless it belongs to the HARA class. Moreover consider the case where we allow $B < 0$, such as the case with quadratic utility ($B = -1$). Since $A + Bx > 0$, we obviously must have $A > 0$. In this case, we have increasing absolute risk aversion. Now the term $t\frac{A}{B}$ in (9) above is negative; but due to the increasing risk aversion, it follows from the above reasoning that we once again have an increase in the investment in the risky asset, $a*$.

But what happens if $A < 0$?[5] Since we must have $B > 0$, we once again have DARA. However, in this case $t\frac{A}{B} < 0$. Thus, it follows from (9) and step (2) above that investment in the risky asset will actually decrease with the wealth tax. Thus we see that for DARA, $B > 0$, the effect of a wealth tax depends critically on whether $A > 0$, $A = 0$, or $A < 0$. These results illustrate a Proposition of Stiglitz (1969, Proposition 1a) that a proportional wealth tax will increase [not change; decrease] investment in the risky asset if we have increasing [constant; decreasing] relative risk aversion.

Suppose now that instead of a wealth tax, we have a flat rate income tax, with the same tax rate t. We do assume, as in Domar and Musgrave (1944) and in Mossin (1968b), that there is a full loss offset for losses. In this case,

[5] Note that, for HARA preferences $A < 0$ corresponds to preferences exhibiting decreasing relative risk aversion, whereas $A > 0$ corresponds to increasing relative risk aversion.

since the tax is only on earned income and not on the initial wealth w as well, it follows from (8) that we can write after-tax wealth as simply refunding the tax on initial wealth in equation (9)

$$\widetilde{x}(1-t) + tw = (\widetilde{x} + \widetilde{z}) + t(\frac{A}{B} + w). \tag{10}$$

We see from (10) that after-tax wealth differs from the case of a wealth tax only by the positive constant tw. Suppose that we once again restrict $B > 0$, so that we have DARA. If we also have $A > 0$, it follows that $a*$ will be higher than it would be with no tax. Indeed, the investment in the risky asset is even higher than it would be in the case where t was a tax on total wealth, not just on income. But reconsider now the case where $A < 0$. Since $A + Bw > 0$ by assumption, the term $t(\frac{A}{B} + w)$ must be positive. Hence investment in the risky asset will increase. Even though a flat-rate wealth tax of t would lessen the investor's investment in the risky asset, a flat-rate income tax of t would increase such investment.

For the case where $B < 0$, with increasing absolute risk aversion, whether $a*$ is lower or higher than with no tax once again depends upon the sign of $t(\frac{A}{B} + w)$. But since we restrict $A + Bx > 0$ for all x, the term $t(\frac{A}{B} + w)$ must be negative when $B < 0$. Hence, a flat-rate income tax will cause $a*$ to rise in this case as well, compared to the case with no tax. However, increasing absolute risk aversion implies that the extra investment in the risky asset will be less with the income tax than it would with a proportional wealth tax.

Note that unlike Mossin, we did not assume that the riskfree rate was zero. Our income tax thus applies not only to returns above a riskfree return, but rather to the riskfree interest as well. Although with an unspecified utility representation, the tax on the riskfree interest can cause problems, as described by Sandmo (2010), our specification that $A + Bx > 0$ resolves such issues in the current setting.

Next, we show that more is invested in the risky asset even when the tax rate is random. To this end, let $\widetilde{\varepsilon}$ be a zero-mean random variable that is statistically independent from portfolio returns. For any realized value of ε, $t + \varepsilon$ denotes the tax rate. We further assume that $0 < t + \widetilde{\varepsilon} < 1$ with probability one. The problem facing the investor is that she must allocate her wealth between the risky and the riskfree asset prior to observing the realization of $\widetilde{\varepsilon}$. In this case, it follows from the above analysis that more wealth is invested in the risky asset than would be invested with no taxes. To see this, consider the objective of the investor without taxes, which is to choose a to maximize expected utility :

$$\max Eu(\widetilde{x}(a)) \equiv Eu(wR_f + a(\widetilde{R} - R_f)) = \int_0^\infty u(wR_f + a(R - R_f))dF(R), \tag{11}$$

where F denotes the distribution function for risky returns.

For a fixed tax rate $t + \varepsilon$ define $U(a, \varepsilon) \equiv Eu(\widetilde{x}(a)(1 - (t + \varepsilon)) + tw)$ as the investor's expected utility, net of her flat-rate income tax, for a given a and a given ε. The first-order condition for the optimal investment at this fixed tax rate is

$$\frac{\partial U(a, \varepsilon)}{\partial a} = 0, \tag{12}$$

which we assume to be satisfied at investment level a_ε. It is easy to show that $U(a, \varepsilon)$ is concave in a for all values of a – not just at the optimal $a*$ From our analysis above, we know that for any ε such that $0 < t + \varepsilon < 1$ we must have $a_\varepsilon > a_N$, where a_N denotes the optimal investment in the risky asset when there are no taxes. This in turn implies that $\frac{\partial U(a,\varepsilon)}{\partial a} > 0$ when evaluated at a_N for every ε, due to the concavity of $U(a, \varepsilon)$.

For a random tax rate $t + \widetilde{\varepsilon}$, the first-order condition for portfolio choice becomes

$$\int_{-\infty}^{+\infty} \frac{\partial U(a, \varepsilon)}{\partial a} dG(\varepsilon) = 0, \tag{13}$$

where G denotes the distribution function for $\widetilde{\varepsilon}$. Let $a*$ denote the solution to (13). We cannot have $a* \leq a_N$, since this would imply that $\frac{\partial U(a,\varepsilon)}{\partial a} > 0$ for every ε, so that (13) cannot hold. Hence, more is invested in the risky asset with the random income tax than would be invested with no tax, $a* > a_N$.

Finally, we note that the methodology used above for a random income-tax rate would also apply for a random wealth tax, so long as we maintain our assumption that $0 < t + \varepsilon < 1$. Indeed, one can easily see how other similar scenarios are possible. For example, this methodology will also apply if we have random inflation that is independent of risky-asset returns. We just need to use a price deflator in place of a proportional wealth-tax rate (as long as we allow only for inflation, with no chance of deflation, i.e. as long as the price deflator remains larger than unity). This allows us to conclude under HARA that, compared to the case with no inflation, a random rate of inflation will cause the investor to increase [not change; decrease] investment in the risky asset if we have increasing [constant; decreasing] relative risk aversion.[6]

4 Portfolio Choice and Risk Vulnerability

We now consider the risk vulnerability model in Gollier and Pratt (1996). They consider a background risk $\widetilde{y}$ with a non-positive mean that is independent of random wealth. They examine conditions under which this background induces less risk taking behavior. In the context of our portfolio problem, this would imply reducing investment in the risky asset. Although the conditions on preference that are equivalent to inducing this type of behavior are quite strong, Gollier and Pratt also present two sufficient conditions for this behavior, both of which are satisfied by HARA utility whenever $B > 0$.[7] However, even these sufficient conditions are not particularly easy to interpret in terms of economic intuition. By choosing an appropriate RTN background risk, we show below that the Gollier-Pratt independent background risk is larger in the low-return states and smaller in the high-return states of nature. The intuition as to why investment in the risky asset decreases then becomes apparent.

To set up this argument, note that the first-order condition to (11) under

[6] Of course, this is a very simplified model of inflation. See, for example, Brennan and Xia (2002) for a more complex model, similar in spirit to the result shown here.

[7] One sufficient condition for risk vulnerability is that preferences satisfy "standard risk aversion," as defined by Kimball (1993), which is characterized by decreasing absolute risk aversion and decreasing absolute prudence. That is, both absolute risk aversion $\frac{-u''(x)}{u'(x)}$ and absolute prudence $\frac{-u'''(x)}{u''(x)}$ are decreasing in x. Another sufficient condition is that absolute risk aversion is both decreasing and convex in wealth.

HARA can be written as follows:

$$\begin{gathered} 0 = E[(A + B\widetilde{x}(a))^{-\frac{1}{B}}(\widetilde{R} - R_f)] \\ \equiv \int_0^{R_f} (A + Bx)^{-\frac{1}{B}}(R - R_f)dF(R) + \int_{R_f}^{\infty} (A + Bx)^{-\frac{1}{B}}(R - R_f)dF(R) \end{gathered}, \tag{14}$$

where $x(a) \equiv wR_f + a(R - R_f)$. We assume that (14) is satisfied at some positive finite value $a*$. Note that the first integral on the left-hand side in (14) is negative, representing the marginal utility cost of a higher a when returns are low, whereas the second term is positive, representing the marginal utility benefit of a higher a when returns are high. The sufficient second-order condition for a maximum is trivially satisfied, since expected utility is concave in a.

If we add any RTN background risk of the form (3), since $\widetilde{\varepsilon}$ is independent of R, the first-order condition becomes

$$\begin{gathered} 0 = E(1 + k + \widetilde{\varepsilon})^{1-\frac{1}{B}} E[(A + B\widetilde{x})^{-\frac{1}{B}}(\widetilde{R} - R_f)] \\ = E[(1 + k + \widetilde{\varepsilon})(1 + k + \widetilde{\varepsilon})^{-\frac{1}{B}}]E[(A + B\widetilde{x})^{-\frac{1}{B}}(\widetilde{R} - R_f)] \\ = [1 + k + \frac{cov(\widetilde{\varepsilon},\ (1+k+\widetilde{\varepsilon})^{-\frac{1}{B}})}{E(1+k+\widetilde{\varepsilon})^{-\frac{1}{B}}}]E(1 + k + \widetilde{\varepsilon})^{-\frac{1}{B}} E[(A + B\widetilde{x})^{-\frac{1}{B}}(\widetilde{R} - R_f)] \\ = [1 + k + \frac{cov(\widetilde{\varepsilon},\ (1+k+\widetilde{\varepsilon})^{-\frac{1}{B}})}{E(1+k+\widetilde{\varepsilon})^{-\frac{1}{B}}}]E[(A + B(\widetilde{x} + \widetilde{z}))^{-\frac{1}{B}}(\widetilde{R} - R_f)]. \end{gathered} \tag{15}$$

Our assumption that $1 + k + \varepsilon > 0$ implies that the constant first term on the left-hand side of the last line in (15) must be positive. Hence, we can re-write the first-order condition as

$$\begin{gathered} \int_0^{R_f} \int_{-\infty}^{+\infty} ((A + B(x + z))^{-\frac{1}{B}}(R - R_f)dG(\varepsilon)dF(R) \\ + \int_{R_f}^{\infty} \int_{-\infty}^{+\infty} ((A + B(x + z))^{-\frac{1}{B}}(R - R_f)dG(\varepsilon)dF(R) = 0, \end{gathered} \tag{16}$$

where G once again denotes the distribution function for $\widetilde{\varepsilon}$. From our previous argument about RTN background risk, this yields the same optimal investment in the risky asset, $a*$ as would hold with no background risk from (14).

To examine the risk-vulnerability result of Gollier and Pratt (1996), consider their independent non-positive mean background risk $\widetilde{y}$. Note that we can define $\widetilde{\varepsilon}$ independent of $\widetilde{R}$ with $E\widetilde{\varepsilon} = 0$ and define $k \leq 0$ implicitly via

$$\widetilde{y} = (k + \widetilde{\varepsilon}) \left[\frac{A}{B} + wR_f \right]. \tag{17}$$

Using this $\widetilde{\varepsilon}$ and this k in (3), so that $\widetilde{z} = (k + \widetilde{\varepsilon})\left[\frac{A}{B} + x\right]$, we see that $\widetilde{z}$ simply replaces wR_f with x when compared to $\widetilde{y}$. Thinking of $(k + \widetilde{\varepsilon})$ as noise, we see that $\widetilde{y}$ has more noise than $\widetilde{z}$ when returns are low $(R < R_f)$ and that $\widetilde{y}$ has less noise than $\widetilde{z}$ when returns are high $(R > R_f)$.

Since we assume $B > 0$, it follows that

$$\int_{-\infty}^{+\infty} ((A + B(x+y))^{-\frac{1}{B}} dG(\varepsilon) > [<] \int_{-\infty}^{+\infty} ((A + B(x+z))^{-\frac{1}{B}} dG(\varepsilon)$$

for each $R < [>] R_f$. This follows since u' is decreasing and convex. The fact that u' is decreasing allows us compare the deterministic parts of $\widetilde{y}$ and $\widetilde{z}$, i.e. to compare the terms with k. The fact that u' is convex allows us to use Jensen's inequality to compare the stochastic parts of $\widetilde{y}$ and $\widetilde{z}$, i.e. to compare the terms containing $\widetilde{\varepsilon}$. To see this more clearly, note that the multiplicative factor on the $\widetilde{\varepsilon}$ term is either wR_f or wealth x; and since $a*$ is positive, the latter term is larger if and only if $R > R_f$.

Calculating $\frac{dEu}{da}$ as in (16) but with y replacing z, it follows that the negative term (marginal costs) is more negative and the positive term (marginal benefit) is less positive, when evaluated at $a*$. Hence, the optimal level of investment in the risky asset will fall, as expected. In other words, compared to its risk-taking-neutral counterpart, which changes marginal utility by a proportional amount everywhere, the independent background risk $\widetilde{y}$ increases marginal utility (and hence increases marginal costs) when returns are low and it decreases marginal utility (and hence decreases marginal benefits) when returns are high.

5 Mossin's Partial Myopia

Mossin (1968b) considers a simple two-period dynamic portfolio problem under HARA preferences. The investor decides at date $t = 0$ how to invest her wealth in a portfolio. At the end of the first period, at date $t = 1$, the investor can than optimally reinvest her realized wealth in a risky asset and/or a riskfree asset. At the end of the second period, at date $t = 2$, the investor then realizes and consumes her final wealth. We assume that returns on the risky asset are statistically independent in each period and that riskfree rate a date $t = 1$, i.e. R_f, is known by the investor at date $t = 0$.

The standard approach to solving such a problem requires a method such as dynamic programming and solving the program "backwards" in time. However, Mossin shows that the first-period investment decision can actually be solved assuming that one hundred percent of wealth will be invested in the riskfree asset at date $t = 1$, which Mossin's calls "partial myopia." Of course, in the special case where $R_f = 1$ at date $t = 1$, such as assumed in Gollier (2001), we then have complete myopia: the investment in the first period is the same as if no second period investment was available.

To establish Mossin's result using RTN background risk, we require the following Lemma, which is a well-known result and is proven, for example, in Gollier(2001, p. 58).

Lemma: *Consider the solution to the standard portfolio problem (11) when preferences exhibit HARA. Let $\widehat{a}$ denote the solution to*

$$E[(1 + B\widehat{a}(\widetilde{R} - R_f))^{-\frac{1}{B}}(\widetilde{R} - R_f)] = 0.$$

Then the solution to (11) satisfies $a* = \widehat{a}(A + BwR_f)$.

We can now prove Mossin's result.

Theorem (Mossin): *Consider the two-period investment problem under HARA utility. At date $t = 0$ the investor chooses an investment in the risky asset that is identical to the one she would choose if all wealth at date $t = 1$ was invested in the riskfree asset.*

Proof: Suppose the investor chooses her investment in the risky asset, a_0, at date $t = 0$ under the assumption that all wealth is re-invested at the riskfree rate. Let $\widetilde{w}_1$ be a random variable denoting her wealth at date $t = 1$ under this investment strategy. Now, consider a change in her re-investment strategy to account for the opportunity to invest in a risky asset at date $t = 1$. We let $\widetilde{R}_2$ denote the risky return for this risky asset and assume it is independent from the distribution of first-period returns, i.e. we assume that $\widetilde{w}_1$ and $\widetilde{R}_2$ are statistically independent random variables.

From the Lemma, it follows that the optimal re-investment in the risky asset at date $t = 1$ is $a_1 = \widehat{a}(A + Bw_1 R_f)$ for any realized wealth w_1. Viewed at date $t = 0$, the investor's random wealth at date $t = 2$ is thus

$$\begin{aligned} \widetilde{w}_2 &= \widetilde{w}_1 R_f + a_1(\widetilde{R}_2 - R_f) \\ &= \widetilde{w}_1 R_f + \widehat{a}(A + B\widetilde{w}_1 R_f)(\widetilde{R}_2 - R_f). \end{aligned} \tag{18}$$

Compared to investing all proceeds in the riskfree asset, the additional risk for this re-investment strategy is thus $[\widehat{a}(\widetilde{R}_2 - R_f)](A + B\widetilde{w}_1 R_f)$. Since $\widetilde{w}_1$ and $\widetilde{R}_2$ are independent, this additional risk is easily seen to be a RTN background risk of the form $\widetilde{z}(x) = (k + \widetilde{\varepsilon})(\frac{A}{B} + x)$, with $x \equiv w_1 R_f$, $\widetilde{\varepsilon} \equiv \widehat{a}B(\widetilde{R}_2 - E\widetilde{R}_2)$ and $k \equiv \widehat{a}B(E\widetilde{R}_2 - R_f)$. Hence, at date $t = 0$, maintaining the strategy of investing a_0 in the risky asset is still optimal. *Q.E.D.*[8]

It is interesting to note that we have no background risks in Mossin's setting. Rather, we viewed $\widetilde{w}_1 R_f$ as being optimal when future investment was all riskfree, and then simply observed that allowing for future risky investment looked no different than adding a RTN background risk.

6 Concluding Remarks

We defined a parametric class of background risks for HARA utility that we call the class of risk-taking-neutral (RTN) background risks. These background risks affect overall satisfaction, but do not alter economic choices. In some cases, such as Mossin's Theorem on partial myopia, we were able to show the result by directly linking wealth to a particular RTN background risk. In other cases, we were able to choose a particular RTN, and then easily compare an extant background risk to our chosen RTN counterpart.

We should also say a word here about the limiting cases under HARA. If $B = 0$ or $B = 1$, the HARA utility as given in (1) is not well defined. However, standard limit arguments easily show that these two cases approach constant absolute risk aversion utility and log utility respectively.[9] As shown

[8] Note that for the special case of constant relative risk aversion (CRRA), which requires $A = 0$, we obtain complete myopia. This follows easily from the Lemma since a_1 is then simply a multiple of w_1, so that a_1/w_1 is constant, as is well known from Merton (1971).

[9] See, for example, Ingersoll (1987).

by Cass and Stiglitz (1970), these two cases, together with the HARA class of utility, play an important role in two-fund separation for portfolio choice models.[10] For CARA utility, we let $\widetilde{z}(x) = k + \widetilde{\varepsilon}$ for all x.[11] Define $u(x) = -\exp(-\theta x)$ for $\theta = A^{-1}$.[12] The indirect utility (4) in this case becomes

$$v(x) = Eu(x + \widetilde{z}) = [E\exp(-\theta(k + \widetilde{\varepsilon}))] \cdot u(x). \tag{19}$$

This background risk is thus risk-taking neutral. This is hardly surprising. Adding or subtracting any constant, be it deterministic (k) or random ($\widetilde{\varepsilon}$) will not affect the decisions made under CARA. The above analysis only shows that such a background risk can be constructed as an example similar to our RTN class.

For the case of logarithmic utility, we let $A = 0$ and $B \to 1$ and define $\widetilde{z}(x) = (k + \widetilde{\varepsilon})x$. The indirect utility (4) is thus

$$v(x) = Eu(x + \widetilde{z}) = E\ln(1 + k + \widetilde{\varepsilon}) + u(x). \tag{20}$$

Of course, $v(x)$ is no longer a positive multiple of $u(x)$ for this case. However, $v(\cdot)$ is still an affine transformation of $u(\cdot)$, so that decisions on the primary risk are unaffected. Likewise, whether the individual is better off or worse off with the RTN background risk depends upon whether $E\ln(1+k+\widetilde{\varepsilon}) > 0$ or $E\ln(1 + k + \widetilde{\varepsilon}) < 0$. Hence, this background risk also exhibits our RTN properties.

Unfortunately, these extensions do not always lend themselves to the types of manipulations we do in this paper. For the case of log utility, the structure is mostly the same, so that our results typically still apply. But the case of CARA utility is oftentimes quite different, without similar results. Since our main objective was to show how our RTN can be used as a benchmark, we do not further explore when the methodology can or cannot be extended to these limits of the HARA class of utility.

Finally, we also note that the RTN class defined in this paper is not exclusively all the background risks with no effect on decision making. As one

[10] These results are also referred to as "mutual fund theorems" in much of the finance literature.

[11] Obviously, letting $B \to 0$ in (3) is not well defined. Instead, we can simply let the "x" term vanish in (3), which gives $\widetilde{z}$ as defined here.

[12] Our assumption that $A + Bx > 0$ implies a positive value for A in this case.

example consider our RTN background risk for quadratic utility ($B = -1$), which gives a multiplicative affine transformation. But with quadratic utility, an independent zero-mean background risk yields an additive dead weight loss to expected utility. Thus it also does not affect economic decisions made on $\widetilde{x}$. The point is that our RTN class of background risks is not an exclusive set of background risks that yield no effect on decisions.

Still, we hope that the RTN class of background risks as used in this paper can prove useful in deriving many new results, as well as better interpreting the intuition of many extant results.

References

[1] Brennan, Michael J. and Xia, Yihong. "Dynamic Asset Allocation under Inflation," *Journal of Finance* 57, pp. 1201-1238.

[2] Campbell, John and Viceira, Luis. *Strategic Asset Allocation*, 2002, Oxford University Press: Oxford, UK.

[3] Cass, David and Stiglitz, Joseph. "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation," *Journal of Economic Theory* 2, pp. 122-160.

[4] Dana, Rose-Anne and Scarsini, Marco. "Optimal Risk Sharing with Background Risk," *Journal of Economic Theory* 133, pp. 152-176.

[5] Doherty, Neil and Schlesinger, Harris. "Optimal Insurance in Incomplete Markets," *Journal of Political Economy* 91, 1983, pp. 1045-1054.

[6] Domar, Evsey and Musgrave, Richard. "Proportional Income Taxation and Risk Taking," *Quarterly Journal of Economics* 58, 1944, pp. 388-422.

[7] Franke, Guenter; Schlesinger, Harris and Stapleton, Richard. "Multiplicative Background Risk," *Management Science* 52, 2006, pp. 146-153.

[8] Franke, Guenter; Schlesinger, Harris and Stapleton, Richard. "Risk Taking with Addititive and Multiplicative Background Risk," *Journal of Economic Theory*, in press.

[9] Gollier, Christian. *The Economics of Risk and Time*. Cambridge: MIT Press, 2001.

[10] Gollier, Christian, and Pratt, John W. "Risk Vulnerability and the Tempering Effect of Background Risk." *Econometrica*, 64, 1996, pp. 1109-24.

[11] Ingersoll, Jonathan E. *Theory of Financial Decision Making*. New Jersey: Rowman & Littlefield, 1987.

[12] Kihlstrom, Richard; Romer, David and Williams, Steve. "Risk Aversion with Random Initial Wealth", *Econometrica* 49, 1981, pp. 911-920.

[13] Kimball, Miles S. “Standard Risk Aversion.” *Econometrica* 61, 1993, pp. 589-611.

[14] Merton, Robert. ”Optimal Consumption and Portfolio Rules in a Continuous-Time Model,” *Journal of Economic Theory* 3, 1971, pp. 373-413.

[15] Mossin, Jan. ”Optimal Multiperiod Portfolio Policies,” *Journal of Business* 41, 1968a, pp. 215-229.

[16] Mossin, Jan. ”Taxation and Risk-Taking: An Expected-Utility Approach,” *Economics* 35, 1968b, pp. 74-82.

[17] Nachman, David. ”Preservation of ’More Risk Averse’ under Expectations.” *Journal of Economic Theory* 28, 1982, pp. 361-368

[18] Ross, Stephen A. “Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications,” *Econometrica* 49, 1981, pp. 621-638.

[19] Sandmo, Agnar. ”Uncertainty in the Theory of Public Finance,” *Geneva Risk and Insurance Review* 35, 2010, pp. 1-18.

[20] Stiglitz, Joseph. ”The Effects of Income, Wealth and Capital Gains Taxation on Risk Taking,” *Quarterly Journal of Economics* 83, 1969, pp. 262-283.