Foundations of Finance Theory: Course Notes

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1 Mean-variance portfolio analysis and the CAPM

Portfolio Demand and the CAPM

Assume

- $j = 1, 2, ..., J$ firms in the economy, with N_j shares outstanding. 100% equity. Cash flow, x_j at time $t = 1$

- $i = 1, 2, ..., I$ investors with wealth $w_{0,i}$ at $t = 0$ and utility

$$u_i = u_i[E(w_i), var(w_i)],$$

  where $w_i \equiv w_{1,i}$ is wealth at time $t = 1$

- Investors have homogeneous expectations

- Investors can borrow or lend at risk-free rate of interest r_f

- No taxes, transactions costs

Definitions

Let:

- $S_j = s_j N_j$ be market capitalisation of j, s_j is price of 1 share

- $n_{i,j}$ shares bought by investor i in firm j

-
$$\alpha_{i,j} = \frac{n_{i,j}}{N_j}$$

- B_i is bond investment of investor i

- $\sigma_{j,j} = var(x_j)$

- $\sigma_{j,k} = covar(x_j, x_k)$

  Hence

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f) \quad (1)$$

  where

$$B_i = w_{0,i} - \sum_j \alpha_{i,j} S_j$$

Portfolio Demand

Investor i chooses shares $n_{i,j}$ to maximise utility:

$$max\ u_i = u_i[E(w_i), var(w_i)] \quad (2)$$

where

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f)$$

subject to

$$B_i + \sum_j \alpha_{i,j} S_j = w_{0,i}$$

Step 1: Compute Portfolio Mean and Variance

Mean:

$$E(w_i) = w_{0,i}(1+r_f) + \sum_j \alpha_{i,j}[E(x_j) - S_j(1+r_f)] \tag{3}$$

Variance:

$$
\begin{aligned}
var(w_i) &= var(\sum_j \alpha_{i,j} x_j) \\
&= \alpha_{i,1}^2 \sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + \dots \\
&+ 2\alpha_{i,1}\alpha_{i,j}\sigma_{1,j} + \alpha_{i,2}^2\sigma_{2,2} + \dots \\
&+ \alpha_{i,j}^2\sigma_{j,j} \tag{4}
\end{aligned}
$$

Hence

$$\frac{\partial E(w_i)}{\partial \alpha_{i,j}} = E(x_j) - S_j(1+r_f) \tag{5}$$

$$\frac{\partial var(w_i)}{\partial \alpha_{i,j}} = 2\alpha_{i,1}\sigma_{j,1} + 2\alpha_{i,2}\sigma_{j,2} + \dots + 2\alpha_{i,j}\sigma_{j,j}$$

$$+ \dots \tag{6}$$

Step 2: First Order Conditions

Hence

$$\frac{\partial u_i}{\partial \alpha_{i,j}} = \frac{\partial u_i}{\partial E(w_i)}\frac{\partial E(w_i)}{\partial \alpha_{i,j}}$$
$$+ \frac{\partial u_i}{\partial var(w_i)}\frac{\partial var(w_i)}{\partial \alpha_{i,j}} = 0, \forall j$$

or

$$-\frac{\frac{\partial u_i}{\partial E(w_i)}}{\frac{\partial u_i}{\partial var(w_i)}}\frac{\partial E(w_i)}{\partial \alpha_{i,j}} = \frac{\partial var(w_i)}{\partial \alpha_{i,j}}, \forall j$$

Substituting (5) and (6)

$$\lambda_i[E(x_j) - S_j(1 + r_f)] = \alpha_{i,1}\sigma_{j,1} + \alpha_{i,2}\sigma_{j,2} + ...$$
$$+ \alpha_{i,j}\sigma_{j,j} + ..., \forall j$$

where

$$\lambda_i = \frac{-\frac{\partial u_i}{\partial Ew_i}}{2\frac{\partial u_i}{\partial var(w_i)}}$$

Step 3: Optimal Stock Proportions

This set of simultaneous equations can be written in matrix form:

$$\lambda_i[E(x) - S(1 + r_f)] = A\alpha_i$$

where

$$E(x) = \begin{pmatrix} E(x_1) \\ E(x_2) \\ . \\ . \\ E(x_J) \end{pmatrix}, S = \begin{pmatrix} S_1 \\ S_2 \\ . \\ . \\ S_J \end{pmatrix}, \alpha_i = \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ . \\ . \\ \alpha_{i,J} \end{pmatrix}$$

and

$$A = \begin{pmatrix} \sigma_{1,1}, \sigma_{1,2}, ..., \sigma_{1,J} \\ \sigma_{2,1}, \sigma_{2,2}, ..., \sigma_{2,J} \\ \sigma_{3,1}, \sigma_{3,2}, ..., \sigma_{3,J} \\ . \quad . \quad . \\ \sigma_{J,1}, \sigma_{J,2}, ..., \sigma_{J,J} \end{pmatrix}$$

The solution is

$$\alpha_i = \lambda_i A^{-1}[E(x) - S(1 + r_f)] \quad (7)$$

(7) proves the mutual fund *Separation Theorem*

Step 4: Market Equilibrium In equilibrium all stocks must be held, or

$$\sum_i \alpha_{i,j} = 1, \forall j$$

Summing (7) over i

$$\sum_i \alpha_i = \sum_i \lambda_i A^{-1}[E(x) - S(1 + r_f)] = \overline{1},$$

where

$$\overline{1} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$$

Solving for S

$$S = \frac{1}{1 + r_f}\left(E(x) - \frac{A\overline{1}}{\lambda}\right), \qquad (8)$$

where

$$A\overline{1} = \begin{pmatrix} \sigma_{1,1} + \sigma_{1,2} + \sigma_{1,3} + ... + \sigma_{1,J} \\ \sigma_{2,1} + \sigma_{2,2} + \sigma_{2,3} + ... + \sigma_{2,J} \\ . \qquad . \qquad . \\ \sigma_{J,1} + \sigma_{J,2} + \sigma_{J,3} + ... + \sigma_{J,J} \end{pmatrix}$$

Step 5: The CAPM

For stock 1:

$$
\begin{aligned}
\sigma_{1,1} \; + \; \sigma_{1,2} + \sigma_{1,3} + \ldots + \sigma_{1,J} &= cov(x_1, x_1) \\
&+ \; cov(x_1, x_2) + \ldots + cov(x_1, x_J) \\
&= \; cov\left(x_1, \sum_{j=1}^{J} x_j\right) \\
&= \; cov\left(x_1, x_m\right),
\end{aligned}
$$

where

$$
x_m = \sum_{j=1}^{J} x_j
$$

is the *market cash flow.*

$$
S_j = \frac{1}{1 + r_f}\left[Ex_j - \frac{1}{\lambda}cov(x_j, x_m)\right],
$$

$$
(9)
$$

Step 6: The CAPM: Expected Return

Divide (9) by S_j,

$$1 = \frac{1}{1 + r_f} \left[\frac{E(x_j)}{S_j} - \frac{1}{\lambda} cov \left(\frac{x_j}{S_j}, x_m \right) \right],$$

Define the return:

$$r_j = \frac{x_j - S_j}{S_j},$$

$$1 = \frac{1}{1 + r_f} \left[E(r_j) + 1 - \frac{S_m}{\lambda} cov \left(\frac{x_j}{S_j}, \frac{x_m}{S_m} \right) \right],$$

$$E(r_j) = r_f + \frac{S_m var(r_m)}{\lambda} \frac{cov(r_j, r_m)}{var(r_m)},$$

$$E(r_j) = r_f + \lambda' \beta_j.$$

Step 7: The CAPM: Closed Form Solution

CAPM holds also for portfolios of shares

If CAPM holds for the market portfolio:

$$E(r_m) = r_f + \lambda' \beta_m.$$

But, $\beta_m = 1$, hence

$$E(r_m) = r_f + \lambda'.$$

The market risk premium:

$$E(r_m) - r_f = \lambda'.$$

and hence

$$E(r_j) = r_f + [E(r_m) - r_f]\beta_j.$$

Optimal Portfolios: Numerical Example

Mean-variance portfolio analysis and the CAPM

Data: Stocks

j	$E(x_j)$	S_j	σ_j	N_j
1	9	7.5	1.5	150
2	38	34	2	400

$$\rho_{1,2} = 0.5$$

Data: Investors

i	λ_i	$w_{0,i}$
1	0.3	10

$$r_f = 0.05$$

Optimal portfolio

Step 1: Compute portfolio mean and variance

<u>Mean</u>

$$E(x_j) - S_j(1 + r_f)$$

$$j = 1: \quad 9 - 7.5(1.05) = 1.125$$

$$j = 2: \quad 38 - 34(1.05) = 2.3$$

$$E(w_i) = 10(1.05) + \alpha_{i,1}(1.125) + \alpha_{i,2}(2.3)$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,1}} = 1.125$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,2}} = 2.3$$

<u>Variance</u>

$$var(w_i) = \alpha_{i,1}^2 \sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + \alpha_{i,2}^2 \sigma_{2,2}$$

$$\sigma_{1,2} = \sigma_1 \sigma_2 \rho_{1,2} = 1.5(2)(0.5) = 1.5$$

$$\sigma_{1,1} = \sigma_1^2 = 1.5^2 = 2.25$$
$$\sigma_{2,2} = \sigma_2^2 = 2^2 = 4$$

$$var(w_i) = \alpha_{i,1}^2 2.25 + 2\alpha_{i,1}\alpha_{i,2}1.5 + \alpha_{i,2}^2 4$$

$$\frac{\partial var(w_i)}{\partial \alpha_{i,1}} = 2\alpha_{i,1}2.25 + 2\alpha_{i,2}1.5$$

$$\frac{\partial var(w_i)}{\partial \alpha_{i,2}} = 2\alpha_{i,1}1.5 + 2\alpha_{i,2}4$$

Step 2: First order conditions

$$\lambda_i[E(x_j) - S_j(1+r_f)] = \sigma_{j,1}\alpha_{i,1} + \sigma_{j,2}\alpha_{i,2}$$

$$j = 1 : 0.3(1.125) = 2.25\alpha_{i,1} + 1.5\alpha_{i,2}$$
$$j = 2 : 0.3(2.3) = 1.5\alpha_{i,1} + 4\alpha_{i,2}$$

Step 3: Matrix form

$$\lambda_i[E(x) - S(1 + r_f)] = A\alpha_i$$

$$A = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix}$$

$$0.3 \begin{pmatrix} 1.125 \\ 2.3 \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix} \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}$$

$$\begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix} \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}$$

Hence

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 0.5926 & -0.222 \\ -0.222 & 0.333 \end{pmatrix} \begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 0.0467 \\ 0.155 \end{pmatrix}$$

Optimal Share allocation

$$n_{i,1} = \alpha_{i,1}(N_1)$$

$$n_{i,1} = 0.0467(150) = 7$$

$$n_{i,2} = \alpha_{i,2}(N_2)$$

$$n_{i,2} = 0.155(400) = 62$$

Stock price

$$s_1 = 7.5/150 = 0.05$$

$$s_2 = 34/400 = 0.085$$

Investment:

stocks

$$n_1 s_1 + n_2 s_2$$

$$7(0.05) + 62(0.085) = 5.62$$

bonds

$$w - 5.62 = 4.38$$

total 10.00

2 Asset pricing: a complete markets model

Financial Theory

Assets:

Value of Cash Flows

CAPM

Options: Multi-period

Black-Scholes   Models

Forward / Futures

Prices: Assets/Options

A General Model
Initial setup and key assumptions

1. Assume a single period from t to $t + T$.

2. Assume forward parity holds.

3. Assume that there are a finite number of states of the world at time $t + T$

   A state-contingent claim on state i is defined as a security which pays $1 if and only if state i occurs.

4. Assume markets are complete.

5. Assume homogeneous expectations

6. Assume that the price of a portfolio or package of contingent claims is equal to the sum of the prices of the individual state-contingent claims.

1: Single Period Model

0———————————1

t———————————$\bullet$ $t + T$

- Nothing happens between t and $t + T$

- Dividends are paid either at t or at $t + T$

- No trading between t and $t + T$
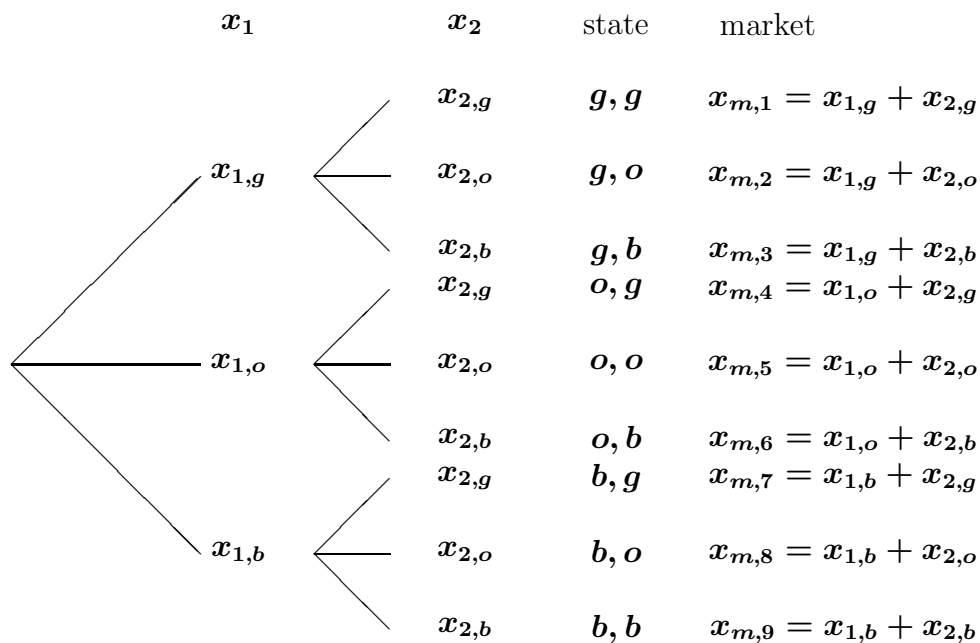
2: Forward Parity

Assume no dividend paid between t and $t + T$

Synthetic forward strategy:

1. **Borrow S_t (price of stock)**

2. **Buy stock**

3. **Repay $S_t(1 + r)$ at $t + T$**

- Net cash flow at t is zero.

- Net cash at $t+T$ is $S_{t+T} - S_t(1+r)$

- $S_t(1+r)$ must be forward price

- $\frac{1}{(1+r)} = B_{t,t+T}$

- $F_{t,t+T} = \frac{S_t}{B_{t,t+T}}$

3: Finite State Space

| x_1		x_2	state	market
		$x_{2,g}$	g,g	$x_{m,1} = x_{1,g} + x_{2,g}$
$x_{1,g}$		$x_{2,o}$	g,o	$x_{m,2} = x_{1,g} + x_{2,o}$
		$x_{2,b}$	g,b	$x_{m,3} = x_{1,g} + x_{2,b}$
		$x_{2,g}$	o,g	$x_{m,4} = x_{1,o} + x_{2,g}$
$x_{1,o}$		$x_{2,o}$	o,o	$x_{m,5} = x_{1,o} + x_{2,o}$
		$x_{2,b}$	o,b	$x_{m,6} = x_{1,o} + x_{2,b}$
		$x_{2,g}$	b,g	$x_{m,7} = x_{1,b} + x_{2,g}$
$x_{1,b}$		$x_{2,o}$	b,o	$x_{m,8} = x_{1,b} + x_{2,o}$
		$x_{2,b}$	b,b	$x_{m,9} = x_{1,b} + x_{2,b}$

4: Complete Market

Buy (or construct) state contingent claims

- Buy an option paying \$1 if $x \geq k_1$
- Sell an option paying \$1 if $x \geq k_2$
- Portfolio pays \$1 if $k_1 < x < k_2$

5: Homogeneous Expectations

$$p_{j,g} = .3 \qquad x_{j,g} \quad 100$$

$$x_{j,o} \quad 90$$

$$p_{j,b} = .5 \qquad x_{j,b} \quad 80$$
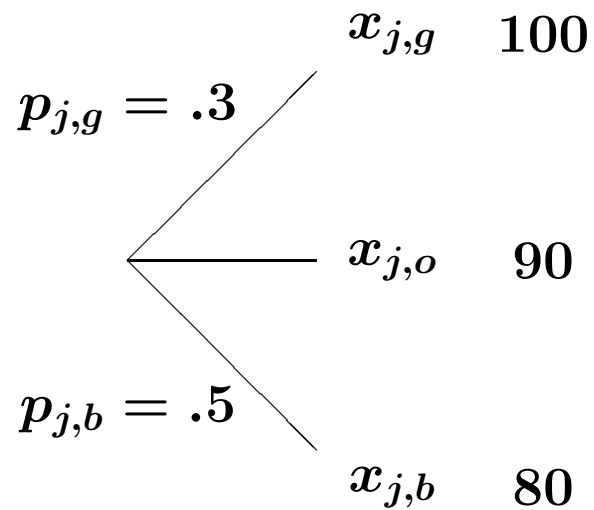
Firm j

- Investors agree on cash flows, x_j
- Investors agree on probabilities, p_j
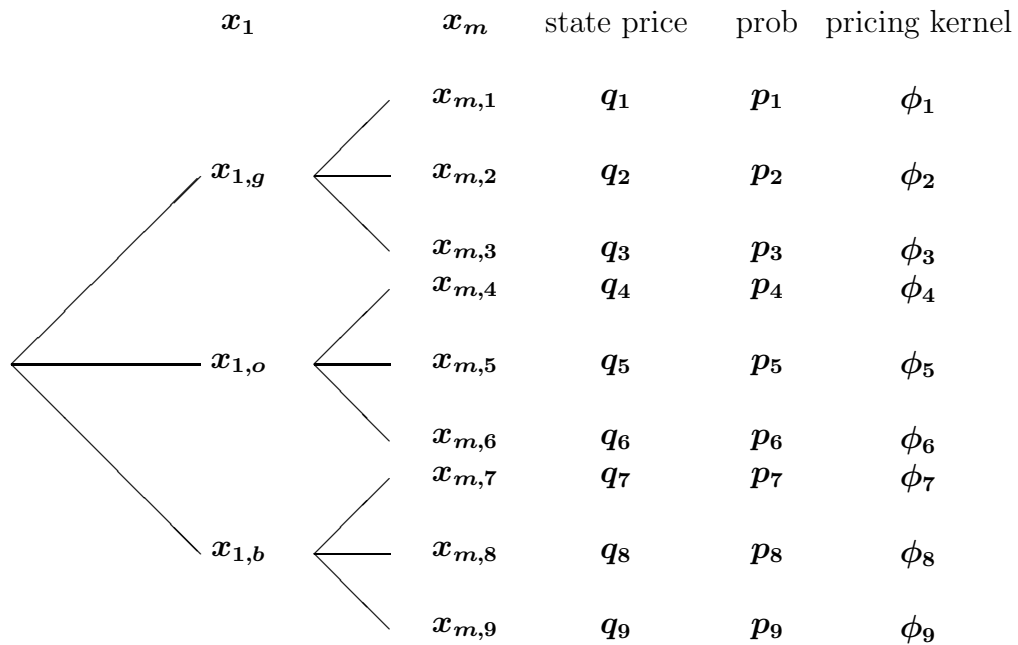
6: Asset Forward Price

Law of One price

$$F_{j,t,t+T} = \sum_i (x_{j,t+T,i} \; q_i)$$

$$F_j = \sum_i (q_i x_{j,i})$$

Properties of the state price

1. The state price, q_i, is always greater than zero.

2. The state prices sum to 1, i.e. $\Sigma_i q_i = 1$.

State Price, Pricing Kernel

x_1		x_m	state price	prob	pricing kernel
		$x_{m,1}$	q_1	p_1	ϕ_1
$x_{1,g}$		$x_{m,2}$	q_2	p_2	ϕ_2
		$x_{m,3}$	q_3	p_3	ϕ_3
		$x_{m,4}$	q_4	p_4	ϕ_4
$x_{1,o}$		$x_{m,5}$	q_5	p_5	ϕ_5
		$x_{m,6}$	q_6	p_6	ϕ_6
		$x_{m,7}$	q_7	p_7	ϕ_7
$x_{1,b}$		$x_{m,8}$	q_8	p_8	ϕ_8
		$x_{m,9}$	q_9	p_9	ϕ_9

$$
\begin{aligned}
F_1 &= x_{1,g}(q_1 + q_2 + q_3) \\
&+ x_{1,o}(q_4 + q_5 + q_6) \\
&+ x_{1,b}(q_7 + q_8 + q_9)
\end{aligned}
$$

The Pricing Kernel, ϕ_i

$$\phi_i = \frac{q_i}{p_i},$$

The properties of ϕ_i are as follows:

1. Since $p_i > 0$ and $q_i > 0$, this means the pricing kernel ϕ_i is a positive function.

2. $E(\phi) = 1$.

Normality of x_m: The CAPM

$$F_j = \sum_i (x_{j,i}\, q_i) = \sum_i p_i\ (\phi_i\ x_{j,i}) = E\left(\phi\ x_j\right)$$

$$F_j = E\left(x_j\phi\right) = E\left(\phi\right) E\left(x_j\right) + cov\left(\phi, x_j\right)$$

$$F_j = E\left(x_j\right) + cov\left(\phi, x_j\right).$$

$$F_j = E\left(x_j\right) + cov\left[\phi(x_m), x_j\right].$$

Covariance and joint probability

Covariance definition

$$cov(x, y) = E\{[x - E(x)][y - E(y)]\}.$$

$$E(x) = p(x_1)x_1 + p(x_2)x_2 + ...$$
$$E(x) = \sum_i p(x_i)x_i$$

$$E(y) = p(y_1)y_1 + p(y_2)y_2 + ...$$
$$E(y) = \sum_i p(y_i)y_i$$

Let $p(x_i, y_j)$ be the joint probability of x_i and y_j occurring, then

$$
\begin{aligned}
cov(x, y) =\ & p(x_1, y_1)[x_1 - E(x)][y_1 - E(y)] \\
& + p(x_1, y_2)[x_1 - E(x)][y_2 - E(y)] \\
& + ... \\
& + p(x_i, y_j)[x_i - E(x)][y_j - E(y)]
\end{aligned}
$$

Stein's Lemma

If x, y are joint normally distributed

$$cov(g(x), y] = E[g'(x)]cov(x, y)$$

Hence

$$cov(x_j, \phi(x_m)] = E[\phi'(x_m)]cov(x_j, x_m)$$

Stein's Lemma

Examples:

1.

$$cov[g(x), y] = E[g'(x)]cov(x, y)$$

let

$$g(x) = a + bx$$
$$g'(x) = b$$
$$cov[g(x), y] = b\, cov(x, y)$$

2. let

$$g(x) = a + bx + cx^2$$
$$g'(x) = b + 2cx$$
$$cov[g(x), y] = [b + 2cE(x)]cov(x, y)$$

3. let

$$g(x) = e^x$$
$$g'(x) = e^x$$
$$cov[g(x), y] = E(e^x)cov(x, y)$$

CAPM: Derivation:
Forward-Cash Flow Version

$$F_j = E\left(x_j\right) E(\phi(x_m)) + cov\left(x_j, \phi(x_m)\right).$$
$$F_j = E\left(x_j\right) E(\phi(x_m)) - \kappa cov\left(x_j, x_m\right).$$
$$= E\left(x_j\right) - \kappa cov\left(x_j, x_m\right),$$

$$\kappa = -E[\phi'(x_m)].$$

Since

$$F_m = E\left(x_m\right) - \kappa cov\left(x_m, x_m\right)$$

$$\kappa = \frac{E(x_m) - F_m}{var(x_m}$$

and

$$F_j = E(x_j) - \frac{E(x_m) - F_m}{var(x_m} cov\left(x_j, x_m\right)$$

CAPM: Applications

1. Cost of capital

- For firm
- For capital project

2. Valuation of non-listed equity

3. Mergers and acquisitions

For applications see Brealey and Myers, Copeland and Weston

APT Arbitrage Pricing Theory

We assume there are k factors and for factor f_k the factor loading is β_{jk}.

$$x_j = a_j + \sum_{k=1}^{K} \beta_{jk} f_k + \varepsilon_j$$

where ε_j is independent of f_k.

$$cov\left(x_j, \phi(x_m)\right) = \sum_{k=1}^{K} \beta_{jk} cov\left(f_k, \phi(x_m)\right) + cov(\varepsilon_j, \phi$$

1. $\varepsilon_j = 0$

2. $cov\left(\varepsilon_j, \phi(x_m)\right) = 0$

If $\varepsilon_j = 0$ or $cov\left(\varepsilon_j, \phi(x_m)\right) = 0$, then

$$
\begin{aligned}
cov\left(x_j, \phi(x_m)\right) &= \sum_{k=1}^{K} \beta_{jk} cov\left(f_k, \phi(x_m)\right) \\
F_j &= E\left(x_j\right) + \beta_{j1} cov\left(f_1, \phi(x_m)\right) \\
&\quad + \beta_{j2} cov\left(f_2, \phi(x_m)\right) + \cdots
\end{aligned}
$$

The Pricing Kernel:An Equilibrium Model

- Assume a 'representative agent' economy

- Let $w_{t+T,i}$ be the wealth of the investor in state i at time $t+T$.

- Assume that the investor is endowed with investible wealth w_t at time t

- The investor can purchase state-contingent claims.

- Choose a set of state-contingent claims paying $w_{t+T,i}$, given a budget allocation of cash, w_t.

We make the following additional assumptions:

1. The investor maximises the expected value of a utility function $u(w_{t+T})$. Hence the investors problem is:

$$max E\left[u\left(w_{t+T}\right)\right] = \sum_i p_i \ u(w_{t+T,i})$$

subject to

$$\sum_i w_{t+T,i} \ q_i \ B_{t,t+T} = w_t \qquad (10)$$

2. The utility function has the properties $u'(w_{t+T}) > 0$ (non-satiation) and $u''(w_{t+T}) < 0$ (risk aversion).

A two-state example:

$$\max\ p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2})$$

subject to

$$[w_{t+T,1}q_1 + w_{t+T,2}q_2]B_{t,t+T} = w_t$$

Lagrangian:

$$
\begin{aligned}
L\ =\ & p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2}) \\
& + \lambda\{w_t B_{t,t+T}^{-1} - w_{t+T,1}q_1 - w_{t+T,2}q_2\}
\end{aligned}
$$

FOC for a maximum:

$$\frac{\partial L}{\partial w_{t+T,1}} = p_1\ u'(w_{t+T,1}) - q_1\lambda = 0.$$

$$\frac{\partial L}{\partial w_{t+T,2}} = p_2\ u'(w_{t+T,2}) - q_2\lambda = 0.$$

Summing over the states i

$$p_1\, u'(w_{t+T,1}) + p_2\, u'(w_{t+T,2}) = \lambda(q_1 + q_2)$$

or

$$E[u'(w_{t+T})] = \lambda,$$

since $q_1 + q_2 = 1$.

Substituting for λ

$$\frac{p_1\, u'(w_{t+T,1})}{E\left[u'(w_{t+T})\right]} = q_1,$$

$$\frac{p_2\, u'(w_{t+T,2})}{E\left[u'(w_{t+T})\right]} = q_2,$$

and hence the first order conditions are satisfied if

$$\frac{u'(w_{t+T,i})}{E\left[u'(w_{t+T})\right]} = \frac{q_i}{p_i} = \phi_i, \quad \forall i.$$

$$\phi_i = \frac{u'(x_{m,i})}{E\left[u'(x_{m,i})\right]} : \ \forall i.$$

Hence we have

$$\phi = \phi(x_m),$$

Case 1: Risk Neutrality

$$u\left(w_{t+T}\right) = a + b\ w_{t+T},$$

$$u'\left(w_{t+T}\right) = b,$$

$$E[u'\left(w_{t+T}\right)] = b,$$

$$\phi_i\left(w_{t+T,i}\right) = \frac{u'\left(w_{t+T,i}\right)}{E\left[u'\left(w_{t+T}\right)\right]} = 1.$$

Forward Price Under Risk Neutrality:

In this case, the forward price is

$$
\begin{aligned}
F_{t,t+T} &= E\left(\phi\, x\right) \\
&= E\left(1 \cdot x\right) \\
&= E\left(F_{t+T,t+T}\right)
\end{aligned}
$$

Case 2: Utility is Quadratic

Assume utility is given by:

$$
\begin{aligned}
u\left(w_{t+T}\right) &= a + bw_{t+T} + \delta w_{t+T}^2 \\
u'\left(w_{t+T}\right) &= b + 2\delta w_{t+T} \\
\phi\left(w_{t+T}\right) &= \frac{u'\left(w_{t+T}\right)}{E\left[u'\left(w_{t+T}\right)\right]} \\
&= \frac{b + 2\delta w_{t+T}}{b + 2\delta E\left(w_{t+T}\right)} \\
cov\left[\phi\left(w_{t+T}\right), x_{t+T}\right] &= \frac{2\delta\, cov\left(w_{t+T}, x_{t+T}\right)}{b + 2\delta E\left(w_{t+T}\right)}
\end{aligned}
$$

Then

$$
F_{t,t+T} = E\left(x_{t+T}\right) + \kappa\ cov\left(w_{t+T}, x_{t+T}\right),
$$

where

$$
\kappa = \frac{2\delta}{b + 2\delta E\left(w_{t+T}\right)}
$$

Asset specific pricing kernel

For asset j, we can write

$$F_j = E\left(\phi x_j\right)$$
$$= E\left[E\left(\phi \,|x_j\right) \; x_j\right]$$
$$F_j = E\left(\psi_j x_j\right)$$

$\psi_j > 0. \; E(\psi_j) = 1.$

Example:

Figure 1.2

$$\psi_{x_{1,g}} = E(\phi|x_1 = x_{1,g})$$

$$\psi_{x_{1,g}} = p_1\phi_1 + p_2\phi_2 + p_3\phi_3$$

Other Readings:

Huang C-F and R.H. Litzenberger (1988) Foundations for Financial Economics, North Holland, Chapters 1 and 4.

Pliska (1997) Introduction to Mathematical Finance, Blackwell, Chapters 1 and 2.

Cochrane (2001) Asset Pricing, Princeton, chs 1-6

Copeland and Weston (1988), Financial Theory and Corporate Policy, Addison-Wesley.

3 Option pricing and risk-neutral valuation

Options and Contingent Claims

A contingent claim on a cash flow x pays some function $g(x)$. The payoff $g(x)$ is contingent on the payoff x on the underlying asset.

A European-style call option maturing at time $t + T$, on a cash flow x pays

$$g(x) = max[x - k, 0]$$

A European-style put option maturing at time $t + T$, on a cash flow x pays

$$g(x) = max[k - x, 0]$$

The Normal Distribution

If $f(x)$ is normal with mean μ_x, and standard deviation σ_x:

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[x-\mu_x]^2}$$

The standard normal distribution, has mean 0 and standard deviation 1 is

$$n(y) = \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}$$

The cumulative normal distribution function is

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

The cumulative standard normal distribution function is

$$N(y) = \int_{-\infty}^{y} f(u)\,du$$

- If $f(x)$ normal,

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

-

$$E(x > a) = \int_{a}^{\infty} x f(x) dx$$

- The normal distribution is the limit $n \to \infty$ of a binomial distribution with n steps

The Joint Normal Distribution

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}}$$

$$\cdot \; e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y}\right]}$$

- If $f(x, y)$ joint normal, $f(x)$ and $f(y)$ are normal

- If $f(x, y)$ joint normal, the conditional distribution $f(x|y)$ is normal

- The regression of x on y is

$$E(x|y) = \mu_x + \frac{\rho\sigma_x}{\sigma_y}(y - \mu_y)$$

A Simple Option Pricing Model

Assume quadratic utility for the representative investor *and* joint-normal distribution

From ch 1

$$
\begin{aligned}
u\left(x_m\right) &= a + bx_m + \delta x_m^2 \\
\phi\left(x_m\right) &= \frac{u'\left(x_m\right)}{E\left[u'\left(x_m\right)\right]} \\
&= \frac{b + 2\delta x_m}{b + 2\delta E\left(x_m\right)}
\end{aligned}
$$

Hence in this case

$$\phi = A + Bx_m.$$

$$F\left[g\left(x\right)\right] = E\left[g\left(x\right)\right] + cov\left(g\left(x\right), A + Bx_m\right)$$

Using Stein's lemma

$$F\left[g\left(x\right)\right] = E\left[g\left(x\right)\right] + BE\left[g'\left(x\right)\right] cov\left(x, x_m\right)$$

The contingent claim is a call option where $g(x) = \max(x - k, 0)$.

$E[g'(x)] = \text{prob}(x > k)$ and hence

$$F[g(x)] = E[g(x)] + B[\text{prob}(x > k)]\, cov(x, x_m) \tag{11}$$

$$F_m = E[x_m] + B\ var(x_m)$$

$$B = \frac{F_m - E(x_m)}{var(x_m)}$$

An application: Valuation of Corporate debt

Assumptions:

1. Company assets produces cash flow x_j at time $t + T$

2. x_j and x_m are joint normal

3. Bonds outstanding B_{t+T} to be repaid at time $t + T$

4. Equity is call option receives:

$$g(x) = max[x_j - B_{t+T}, 0]$$

   at $t + T$

5. Forward value of equity is

$$F[g(x)] = E[g(x)] + B\,[\mathrm{prob}(x > k)]cov(x_j, x_m)$$

6. Spot value of equity is

$$S_t = F[g(x)]B_{t,t+T}$$

7. Value of company assets is V_t

8. Value of the debt is

$$V_t - S_t$$

The Black-Scholes Model

- **Purpose**

  - To price options on stocks
  - To value corporate liabilities
  - To evaluate credit risk

- **Assumptions**

  - Lognormal asset price
  - No dividends
  - Option is European-style

- **Main Features**

  - Preference-free relationship
  - Mean of asset not required
  - Inputs: asset price, volatility

- **Generalisations**

  - Forex options
  - American-style options

Pricing of Contingent Claims

$$F_{t,t+T}\left[g\left(x_{j,t+T}\right)\right] = \sum_i q_i g\left(x_{j,t+T,i}\right)$$

$$= \sum_i p_i \phi_i g\left(x_{j,t+T,i}\right)$$

$$= E\left[g\left(x_{j,t+T}\right)\phi\right]$$

$$= E\left[g(x_j)E(\phi|x_j)\right]$$

$$= E\left[g(x_j)\psi(x_j)\right]$$

Characteristics of the Pricing Kernel, ψ_j

Assume that

$$\psi_j = \alpha x_j^{\beta}$$

where α and β are constants.

The asset specific pricing kernel has constant elasticity.

The elasticity of the pricing kernel defined by the relationship:

$$\eta = -\frac{\partial \psi_j / \psi_j}{\partial x_j / x_j}$$

Special Case of Options on x_m

Conditions for constant elasticity

- In this case $\psi_j = \phi(x_m)$

- $\phi(x_m)$ has constant elasticity if the representative investor has power utility

- If $u(x_m) = x_m^\gamma$ then ϕ has constant elasticity

The Lognormal Distribution

If $\ln(x)$ is normal with

$$E[\ln(x)] = \mu_x, var[\ln(x)] = \sigma_x^2$$

then x is lognormally distributed

If $f(\ln x)$ is normal with mean μ_x, and standard deviation σ_x:

$$f(\ln x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x - \mu_x]^2}$$

- The lognormal distribution is the limit of a process where $\ln x$ is binomial

- Assume a stock price S_t goes up by a proportion u with probability p or down by d with probability $(1 - p)$ then S_t is log-binomial

Further Properties of Lognormal Variables

1. The expected value of a lognormal variable: Let Y be lognormal, with $y = \ln(Y)$

$$E(Y) = E(e^y) = e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

2. Also,

$$E(Y^b) = E(e^{by}) = e^{b\mu_y + \frac{1}{2}b^2\sigma_y^2}.$$

3. If x, y are lognormal, xy is lognormal.

$$\ln xy = \ln x + \ln y$$

It follows

$$E(\ln xy) = E(\ln x) + E(\ln y)$$

or

$$E(\ln xy) = \mu_x + \mu_y$$

$$var(\ln xy) = var(\ln x) + var(\ln y) \\ + 2cov(\ln x, \ln y)$$

or

$$var(\ln xy) = \sigma_x^2 + \sigma_y^2 + 2\sigma_{x,y}$$

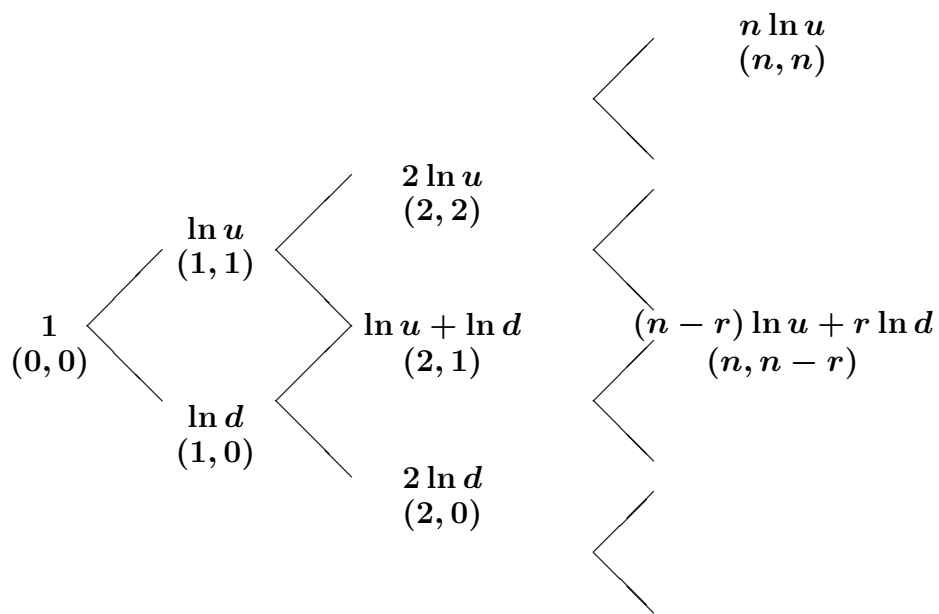## 4. It follows:

$$E(xy) = e^{E(\ln xy) + \frac{1}{2}var(\ln xy)} \\ = e^{\mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_y^2 + \sigma_{x,y}}$$

A General Binomial Process

$$Su^n$$
$$(n,n)$$

$$Su^2$$
$$(2,2)$$

$$Su$$
$$(1,1)$$

$$S$$
$$(0,0)$$

$$Sud$$
$$(2,1)$$

$$Su^{n-r}d^r$$
$$(n,n-r)$$

$$Sd$$
$$(1,0)$$

$$Sd^2$$
$$(2,0)$$

A Log-Binomial Process

$n \ln u$
(n, n)

$2 \ln u$
$(2, 2)$

$\ln u$
$(1, 1)$

1
$(0, 0)$

$\ln u + \ln d$
$(2, 1)$

$(n - r) \ln u + r \ln d$
$(n, n - r)$

$\ln d$
$(1, 0)$

$2 \ln d$
$(2, 0)$
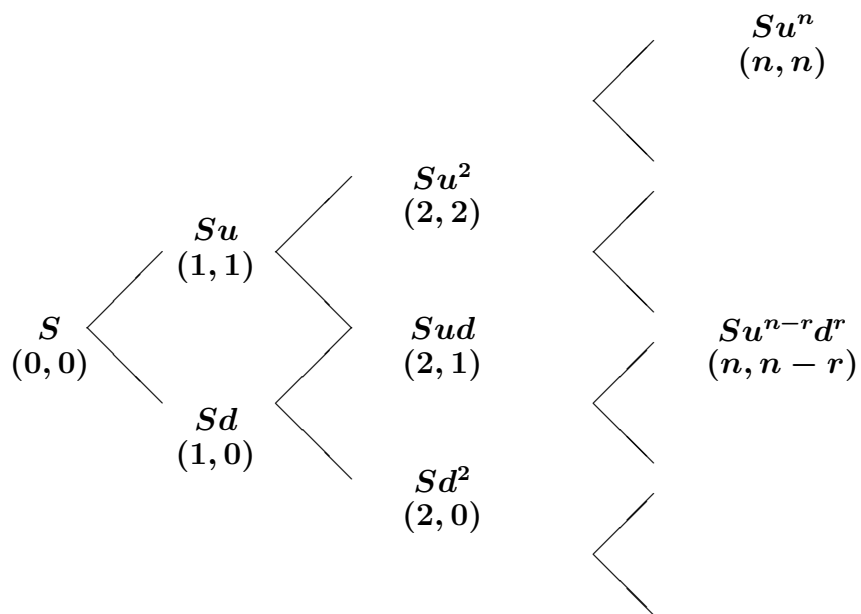
Lognormal x_j and the Black-Scholes Model

Properties of Lognormality

Pricing Kernel

Risk-adjusted PDF

Forward Price of x_j

RNVR

Black-Scholes Model

Module 1: Notation and Lognormal Properties

$$E\left[\ln x\right] = \mu_x$$

$$var\left[\ln x\right] = \sigma_x^2$$

$$f\left(\ln x\right) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x - \mu_x]^2}$$

$$E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E(x_j^\beta) = e^{\beta\mu_x + \frac{1}{2}\beta^2\sigma_x^2}$$

Option payoff:

$$g(x_j) = g\left(e^{\ln x_j}\right) = h(\ln x_j)$$

Module 2: Asset Specific Pricing Kernel

$$
\begin{aligned}
F\left[g\left(x_j\right)\right] &= E\left[g\left(x_j\right)\phi(x_m)\right] \\
&= E\left[g\left(x_j\right)E\left[\phi(x_m)|x_j\right]\right] \\
&= E\left[g\left(x_j\right)\psi(x_j)\right]
\end{aligned}
$$

Assume that

$$
\psi_j = \alpha x_j^{\beta}
$$

where α and β are constants.

Elasticity:

$$
\eta = -\frac{\partial \psi_j / \psi_j}{\partial x_j / x_j}
$$

$$
\eta = -\alpha \beta x_j^{\beta-1}\frac{x_j}{\alpha x_j^{\beta}} = -\beta
$$

$E(\psi_j) = 1$ **implies that**

$$
\alpha = e^{-\beta \mu_x - \frac{1}{2}\beta^2 \sigma_x^2}.
$$

Module 3: Risk-Adjusted PDF

PDF of x_j:

$$f(\ln x_j) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]^2}$$

Risk-adjusted PDF of x_j:

$$\hat{f}(\ln x_j) = f(\ln x)\psi(x_j) = \alpha x_j^\beta \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]}$$

$$\hat{f}(\ln x_j) = e^{-\beta\mu_x - \frac{1}{2}\beta^2\sigma_x^2}x_j^\beta \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]^2}$$

$$\hat{f}(\ln x_j) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\mu_x + \beta\sigma_x^2)]^2}$$

Module 4: Forward Asset Price of x_j

From ch 1:

$$F_j = E[x_j \psi(x_j)]$$

$$F_j = e^{\mu_x + \mu_\psi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\psi^2 + \sigma_{x\psi}}$$

since $E(\psi) = 1$

$$F_j = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x\psi}}$$

$$\ln F_j - \frac{1}{2}\sigma_x^2 = \mu_x + \sigma_{x\psi}$$

Module 5: The Lognormal RNVR

Under Risk Neutrality:

$$E\left[g\left(x_j\right)\right] = E\left[h\left(\ln x_j\right)\right]$$

$$= \int h(\ln x_j) f(\ln x_j) d\ln x_j$$

$$F_j = E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E[g(x_j)] = \int h(\ln x_j)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d\ln$$

Under Risk Aversion

$$F[g(x_j)] = \int h(\ln x_j)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d\ln$$

Module 6: Option Prices: The Black-Scholes Model

$$F[g(x_j)] = E\left[g\left(x_j\right)\psi\left(x_j\right)\right]$$

$$F[g(x_j)] = \int h(\ln x_j)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j-(\ln F-\frac{1}{2}\sigma_x^2)]^2}d\ln$$

The Black-Scholes Model: A Call Option

A European-style call option on x_j, with strike price k has a payoff at time $t+T$:

$$g(x_j) = max(x_j - k, 0).$$

or

$$h(\ln x_j) = max(e^{\ln x_j} - e^{\ln k}, 0).$$

$$F[g(x)] = \int_{-\infty}^{\infty} max\left(e^{\ln x_j} - k, 0\right) \hat{f}(\ln x_j) \, d\ln x_j$$

where

$$\hat{f}(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}\left[\ln x_j - (\ln F_j - \frac{1}{2}\sigma_x^2)\right]^2}$$

$$F[g(x)] = \int_{\ln k}^{\infty} \left(e^{\ln x_j} - k\right) \hat{f}(\ln x_j) \, d\ln x_j$$

$$\begin{aligned}
F[g(x)] = {} & \int_{\ln k}^{\infty} e^{\ln x_j} \hat{f}(\ln x_j) \, d\ln x_j \\
& - k \int_{\ln k}^{\infty} \hat{f}(\ln x_j) \, d\ln x_j.
\end{aligned}$$

The Normal Distribution

Probability $y > a$

and

$$\int_a^\infty f(y)dy.$$

$$\int_a^\infty f(y)dy = 1 - N\left(\frac{a - \mu_y}{\sigma_y}\right) = N\left(\frac{\mu_y - a}{\sigma_y}\right).$$

The Normal Distribution

Expected value of e^y given $y > a$

$$\int_a^\infty e^y f(y) dy$$

$$\int_a^\infty e^y f(y) dy = N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

Evaluate the integrals

$y = \ln x_j$ and $a = \ln k.$

The Normal Distribution

Applying the above results and substituting the mean of $\hat{f}\left(\ln x_j\right)$, $\mu_x = \ln\left(F_j\right) - \frac{\sigma_x^2}{2}$ and $a = \ln k$, we have

$$N\left(\frac{\mu_y - a}{\sigma_y}\right) = N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k}{\sigma_x}\right)$$

and

$$N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2}$$

$$= F_j N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k + \sigma_x^2}{\sigma_x}\right)$$

The Black-Scholes Model: Forward Price

$$F\left[g\left(x\right)\right] = F_j N\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right) - kN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

Asset forward price F_j

The (logarithmic) variance σ_x^2,

The strike price of the option, k.

The Black-Scholes Model: Spot Price

$$S_t[g(x_j)] = B_{t,t+T}F_jN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right)$$
$$- B_{t,t+T}kN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

or, using conventional notation:

$$S_t[g(x_j)] = B_{t,t+T}F_jN\left(d_1\right) - B_{t,t+T}kN\left(d_2\right),$$

where

$$d_1 = \frac{\ln\frac{F_j}{k} + \frac{\sigma_x^2}{2}}{\sigma_x}$$
$$d_2 = d_1 - \sigma_x.$$

The Black-Scholes Model: Applications

1. **Non-dividend paying assets:**

2. **Assets paying a non-stochastic dividend:**

3. **Assets paying a stochastic proportional dividend**

The Black-Scholes Model:
Non-dividend paying assets

In this case, spot-forward parity for the underling asset means that the spot price of the asset is

$$S_t = F_j B_{t,t+T},$$

where $B_{t,t+T} = e^{-rT}$ and r is the continuously compounded interest rate.

$$S_t[g(x_j)] = S_t N\left(d_1\right) - k e^{-rT} N\left(d_2\right), \tag{12}$$

$$d_1 = \frac{\ln \frac{S_t}{k} + rT + \frac{\sigma_x^2}{2}}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x$$

The Black-Scholes Model: Assets paying a non-stochastic dividend

known dividend D_{t+T} at time $t + T$.

In this case, spot-forward parity implies

$$S_t = (F_j + D_{t+T})B_{t,t+T},$$

$$S_t[g(x_j)] = (S_t - D_{t+T}e^{-rT})N(d_1) - ke^{-rT}N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S_t - D_{t+T}e^{-rT}}{k} + \frac{\sigma_x^2}{2} + rT}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x.$$

The Black-Scholes Model: Assets paying a stochastic proportional dividend

$$D_{t+T} = \delta x_j$$

$$S_t = F_j(1 + \delta)B_{t,t+T}.$$

$$S_t[g(x_j)] = \frac{S_t}{(1 + \delta)}N(d_1) - ke^{-rT}N(d_2).$$

where

$$d_1 = \frac{\ln \frac{S_t}{(1+\delta)} + rT - \ln k + \frac{\sigma_x^2}{2}}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x.$$

Spot-Forward Parity

1. Non-dividend paying assets:

$$S_t = F_j B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}}$$

2. Assets paying a non-stochastic dividend:

$$S_t = (F_j + D_{t,t+T}) B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}} - D_{t+T}$$

3. Assets paying a stochastic proportional dividend

$$S_t = F_j (1 + \delta) B_{t,t+T}.$$

$$F_j = \frac{S_t}{B_{t,t+T}} \left(\frac{1}{1 + \delta} \right)$$

The Black-Scholes Model: Extensions

1. Multivariate Options:

2. Compound Options:

3. Bermudan-style options

4. American-style options

5. Non-Constant Elasticity

6. Non-Lognormal Asset Distributions

4 Multi-period asset pricing

Asset prices in a Multi-period Economy

1. Chapter 5: Extension of ch 1 model to multiple periods

2. Chapter 6: Application to analysis of Forward contracts and Futures contracts

Motivation

1. Valuation of a company: more realistic model

2. Capital budgeting: Should an Investment project be accepted

3. Does the CAPM extend and apply in a multi-period world?

4. Basic problem

   - Multiple dates
   - Multiple states at each date

Asset prices in a Multi-period Economy: Introduction

- Valuation under certainty

- Time-state preference approach

- Rational expectations approach

Asset prices in a Multi-period Economy: Certainty Case

$$t+1 \quad t+2 \quad ... \quad t+n$$
$$X_{j,t+1} \quad X_{j,t+2} \quad ... \quad X_{j,t+n}$$
$$B_{t,t+1} \quad B_{t,t+2} \quad ... \quad B_{t,t+n}$$

$$S_{j,t} = B_{t,t+1}(X_{j,t+1} + S_{j,t+1})$$
$$S_{j,t+1} = B_{t+1,t+2}(X_{j,t+2} + S_{j,t+2})$$
$$S_{j,t+2} = B_{t+2,t+3}(X_{j,t+3} + S_{j,t+3})$$
$$... = ...$$
$$S_{j,t+n-1} = B_{t+n-1,t+n}X_{j,t+n}$$

$$S_{j,t} = B_{t,t+1}X_{j,t+1} + B_{t,t+1}B_{t+1,t+2}X_{j,t+2}...$$
$$+ \; B_{t,t+1}B_{t+1,t+2}...B_{t+n-1,t+n}X_{j,t+n}$$

Asset prices in a Multi-period Economy: Basic Set-up

1. Markets are complete. Investors can purchase claims that pay \$1 if and only if a given state occurs at a given point in time.

2. Assume there are just two periods; period $t-t+1$ and period $t+1-t+2$.

3. Value a stock j which pays dividends $x_{j,t+1}$ at time $t+1$ and $x_{j,t+2}$ at time $t+2$.

4. There are $i = 1, 2, \cdots, I$ states at time $t+1$ and there are $k = 1, 2, \cdots, K$ states at time $t+2$.

Asset prices in a Multi-period Economy: Basic Set-up

$$t \qquad t+1 \qquad t+2$$

$$k = 1$$
$$i = 1$$
$$k = 2$$
$$i = 2 \qquad k = 3$$
$$k = 4$$
$$i = 3$$
$$k = 5$$
$$k = 6$$
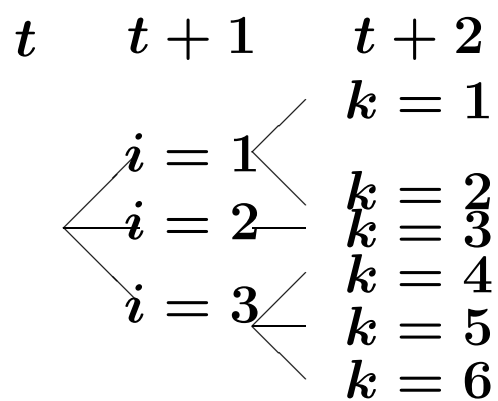
Time-State Preference Approach

1. Treat each period cash flow as separate valuation

2. Define $\phi_{t,t+1} = q_i/p_i$

3. Define $\phi_{t,t+2} = q_k/p_k$
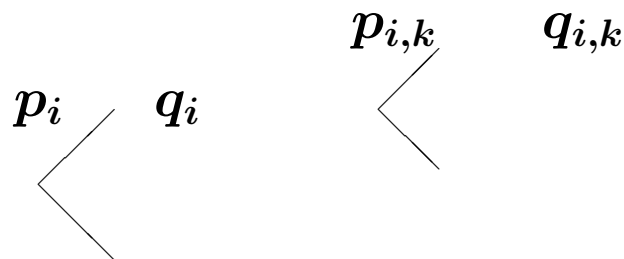
   and so on.

   Then applying ch 1 for each T:

   $$S_t = \sum_{T=1}^{n} F_{t,t+T} B_{t,t+T},$$

   where

   $$F_{t,t+T} = E\left[\phi_{t,t+T} x_{t+T}\right].$$

Rational Expectations Approach: Basic Set-up

Consider the following state diagram:-

$$p_{i,k} \qquad q_{i,k}$$

$$p_i \quad q_i$$

p_i is the probability of state i occuring at time $t + 1$

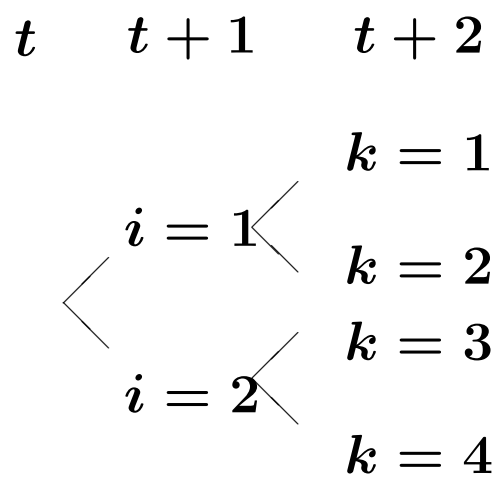$p_{i,k}$ is the conditional probability of state k occuring at $t + 2$

q_i is the forward price of a dollar paid *if and only if* state i occurs.

$q_{i,k}$ is the (conditional) forward price, in state i of a dollar paid *if and only if* state k occurs.

Equivalence of TSP and RE Prices: Example

Let $i = 1, 2$, $k = 1, 2, 3, 4$

Assume a state space as follows:

$$t \qquad t+1 \qquad t+2$$

$$i = 1 \begin{cases} k = 1 \\ k = 2 \end{cases}$$

$$i = 2 \begin{cases} k = 3 \\ k = 4 \end{cases}$$

Equivalence of TSP and RE Prices: Example

1. Zero-coupon bond prices

$$B_{t,t+1} = 0.9$$
$$B_{t+1,t+2,1} = 0.9$$
$$B_{t+1,t+2,2} = 0.8$$

2. Forward state prices

$$q_i, \ i = 1 : \ q_1 = 4/9$$
$$q_i, \ i = 2 : \ q_2 = 5/9$$
$$q_{i,k}, \ i = 1, k = 1 : \ q_{1,1} = 4/9$$
$$q_{i,k}, \ i = 1, k = 2 : \ q_{1,2} = 5/9$$
$$q_{i,k}, \ i = 2, k = 3 : \ q_{2,3} = 1/2$$
$$q_{i,k}, \ i = 2, k = 4 : \ q_{2,4} = 1/2$$

Equivalence of TSP and RE Prices: Example

t	$t+1$	$t+2$
?		1
$(4/9)(0.9).(4/9)(0.9)$	0.4	
	-0.4	1

Note: 0.4 is spot price at date 1, in state 1

Strategy ensures \$1 in state 1 at date 2

Rational Expectations Approach

Using ch 1 method:

$$S_t = B_{t,t+1} \sum_i p_i \phi_{t,t+1,i} \ S_{t+1,i}$$
$$= B_{t,t+1} \ E_t \left(\phi_{t,t+1} S_{t+1} \right)$$

This is the value of x_{t+2} at time t.

Substituting for S_{t+1}:

$$S_t = B_{t,t+1} E_t \left[\phi_{t,t+1} B_{t+1,t+2} \ E_{t+1} \left(\phi_{t+1,t+2} \ x_{t+2} \right) \right]$$

The Fundamental Equation of Valuation

The value of any cash flow $x_{j,t+T}$:

$$S_{j,t} = B_{t,t+1}E_t[\phi_{t,t+1}B_{t+1,t+2}E_{t+1}(\phi_{t+1,t+2}$$
$$.... B_{t+T-1,t+T}E_{t+T-1}(\phi_{t+T-1,t+T}x_{j,t+T}))]$$

where $\phi_{\tau,\tau+1}$ is the period-by-period pricing kernel.

The Relationship Between the Pricing Kernels when Interest Rates are Non-stochastic

The cost of a claim on state k is $B_{t,t+2}q_k$.

Alternatively, cost is

$B_{t,t+1}q_i B_{t+1,t+2}q_{i,k}$.

Hence

$$B_{t,t+2}q_k = B_{t,t+1}q_i B_{t+1,t+2}q_{i,k}.$$

$$B_{t,t+2}\frac{q_k}{p_k} = B_{t,t+1}\frac{q_i}{p_i}B_{t+1,t+2}\frac{q_{i,k}}{p_{i,k}},$$

and hence

$B_{t,t+2}\phi_{t,t+2} = B_{t,t+1}\phi_{t,t+1}B_{t+1,t+2}\phi_{t+1,t+2}$.

But since in this case, $B_{t,t+2} = B_{t,t+1}B_{t+1,t+2}$, we have

$$\phi_{t,t+2} = \phi_{t,t+1}\ \phi_{t+1,t+2}.$$

or, in general

$$\phi_{t,t+T} = \phi_{t,t+1}\ \phi_{t+1,t+2}....\phi_{t+T-1,t+T}.$$

Rational Expectations Approach: Basic Set-up

We now define the period-by-period pricing kernel by the relationship:

$$\phi_{t+1,t+2,i,k} = \frac{q_{i,k}}{p_{i,k}}.$$

Using this we can write:

$$S_{t+1,i} = B_{t+1,t+2,i} \sum_k p_{i,k} \phi_{t+1,t+2,i,k} \ x_{t+2,k}$$
$$= B_{t+1,t+2,i} \ E_{t+1,i} \left(\phi_{t+1,t+2,i} \ x_{t+2} \right)$$

This is the value of x_{t+2} at time $t+1$ in state i.

The Structure of State-Contingent Claim Prices

	Time t	Time $t+1$	Time
Spot price at t of \$1 paid at $t+2$	q_k^*		
Spot price at t of \$1 paid at $t+1$	q_i^*		
Spot price at $t+1$ of \$1 paid at $t+2$		$q_{i,k}^*$	
Period $-$ by $-$ period valuation	$q_i^* q_{i,k}^*$	$q_{i,k}^*$	\$1
One $-$ long $-$ period valuation	q_k^*		\$1

$$q_i^* = q_i B_{t,t+1}$$
$$q_k^* = q_k B_{t,t+2}$$
$$q_{i,k}^* = q_{i,k} B_{t+1,t+2,i}$$

Equivalence of TSP and RE Prices

t	$t+1$	$t+2$
$-q_k B_{t,t+2}$		1
$-q_i B_{t,t+1} q_{i,k} B_{t+1,t+2,i}$	$+q_{i,k}B_{t+1,t+2,i}$ $-q_{i,k}B_{t+1,t+2,i}$	1

t	$t+1$	$t+2$
$-q_k^*$		1
$-q_i^* q_{i,k}^*$	$+q_{i,k}^*$ $-q_{i,k}^*$	1

Rational Expectations Equilibrium: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1}E_t[\phi_{t,t+1}B_{t+1,t+2}E_{t+1}(\phi_{t+1,t+2}x_{j,t+2})]$$

Alternatively, this pricing equation can be written as

$$S_{j,t} = B_{t,t+1}E_t[\phi_{t,t+1}S_{j,t+1}],$$

where

$$S_{j,t+1,i} = B_{t+1,t+2,i}E_{t+1,i}(\phi_{t+1,t+2,i}x_{j,t+2})].$$

From ch 1, the CAPM holds over $t+1$ to $t+2$:

$$
\begin{aligned}
S_{j,t+1,i} = {} & B_{t+1,t+2,i}[E_{t+1,i}(x_{j,t+2}) \\
& - \lambda_{t+1,i}cov_{t+1,i}(x_{j,t+2}, x_{m,t+2})].
\end{aligned}
$$

where $\lambda_{t+1,i}$ is the market price of risk in state i at $t+1$

Multi-period Valuation: Joint-Normal Cash Flows

$$S_{j,t+1,i} = B_{t+1,t+2,i}[E_{t+1,i}(x_{j,t+2})$$
$$- \lambda_{t+1,i}cov_{t+1,i}(x_{j,t+2}, x_{m,t+2})].$$

Now assume:

1. $B_{t+1,t+2,i}$ is non-stochastic

2. $\lambda_{t+1,i}$ is non-stochastic

These assumptions were made by Stapleton and Subrahmanyam *Econometrica* (1978)

Then it follows that:

The value, $S_{j,t+1,i}$, is normally distributed.

Also $S_{m,t+1,i}$, is normally distributed and the conditions for the period-by-period CAPM hold.

Multi-period Valuation: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1}\left[E_t\left(S_{j,t+1}\right) - \lambda_t cov\left(S_{j,t+1}, S_{m,t+1}\right)\right].$$

Substituting for the time $t+1$ prices, we can write:

$$
\begin{aligned}
S_{j,t} =\ & B_{t,t+1}B_{t+1,t+2}E_t(x_{j,t+2}) \\
& - B_{t,t+2}\lambda_t cov_t\left[E_{t+1}(x_{j,t+2}), E_{t+1}(x_{m,t+2}\right] \\
& - B_{t,t+2}\lambda_{t+1} cov_{t+1}\left[x_{j,t+2}, x_{m,t+2}\right]
\end{aligned}
$$

Multi-Period Valuation: Extensions

1. If $B_{t+1,t+2,i}$ is stochastic: Intertemporal CAPM (Merton, Long)

2. Equilibrium model to determine $\phi_{t,t+1}$, ch 5.8

3. Consumption CAPM, see Cochrane

4. RE approach, see Pliska

5. Applications: Futures v Forward prices

5 Forward and futures prices

Forward Contracts and Futures Contracts: Motivation

- Futures contracts are the way most contracts are traded on public exchanges

- In the case of interest-rate related securities the difference between forward and futures prices is important

- Determine futures prices of options (as traded on LIFFE)

- Futures prices are expected values (under risk-neutral measure)

Forward Contracts and Futures Contracts

- A *forward contract* is an agreement made at a point in time t to purchase or sell an asset at a later date $t + T$.

- A *futures contract* is similar to a forward contract but is marked to market on a daily basis as time progresses from t to $t + T$.

Futures Contracts: Payoffs

A long futures contract made at time t, with maturity T, to buy an asset at a price $H_{t,t+T}$ has a payoff $[H_{t+\tau,t+T} - H_{t+\tau-1,t+T}]$ at time $t + \tau$.

Hence the payoff at time $t+1$ is $H_{t+1,t+T} - H_{t,t+T}$

A short futures contract made at time t, with maturity T, to sell an asset at a price $H_{t,t+T}$ has a payoff $[-H_{t+\tau,t+T} + H_{t+\tau-1,t+T}]$ at time $t + \tau$.

Hence the payoff at time $t+1$ is $-H_{t+1,t+T} + H_{t,t+T}$

Forward Contracts: Payoffs

The payoff on a long forward contract is $[S_{t+T} - F_{t,t+T}]$ at time $t + T$,

The payoff on a short forward contract is $[-S_{t+T} + F_{t,t+T}]$ at time $t + T$

Forward and Futures: Payoffs

t	$t+1$	$t+2$	$t+3$	$\cdots$	$t+T$
Futures	$\boldsymbol{H_{t+1,t+T}}$	$\boldsymbol{H_{t+2,t+T}}$	$\boldsymbol{H_{t+3,t+T}}$	$\cdots$	$\boldsymbol{H_{t+T,t+T}}$
contract	$\boldsymbol{-H_{t,t+T}}$	$\boldsymbol{-H_{t+1,t+T}}$	$\boldsymbol{-H_{t+2,t+T}}$	$\cdots$	$\boldsymbol{-H_{t+T-1,t+T}}$
Forward					$\boldsymbol{S_{t+T}}$
contract					$\boldsymbol{-F_{t,t+T}}$

Long Futures and Long Forward

- **Futures is marked-to-market daily**
- **Forward pays off at end of contract**

Characterization of Futures Price

[CIR (1981) Proposition 1]

Consider an asset with a price $\widetilde{S}_{t+T}$ at time $t + T$. The futures price of the asset, $H_{t,t+T}$, is the time t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_{t+T}}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{t+T-1,t+T}}$$

at time $t + T$.

Cox J.C., J. E. Ingersoll Jr and S. A. Ross (1981) The relationship between forward prices and futures prices, *Journal of Financial Economics*, 9, pp.321-346.

A Three-year Example

Strategy

- **Time 0: invest $H_{0,3}$ overnight**
- **Time 0: $\frac{1}{B_{0,1}}$ long futures**
- **Time 1: re-invest proceeds overnight**
- **Time 1: $\frac{1}{B_{0,1}B_{1,2}}$ long futures**
- **Time 2: re-invest proceeds overnight**
- **Time 2: $\frac{1}{B_{0,1}B_{1,2}B_{2,3}}$ long futures**

Characterization of Futures Price

Time	Profits from futures	Gain from investment	Net position
0	$-$	$-$	$H_{0,3}$
1	$\frac{1}{B_{0,1}}[H_{1,3} - H_{0,3}]$	$\frac{1}{B_{0,1}}[H_{0,3}]$	$\frac{1}{B_{0,1}}[H_{1,3}]$
2	$\frac{1}{B_{0,1}B_{1,2}}[H_{2,3} - H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}}[H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}}[H_{2,3}]$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{3,3} - H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{3,3}]$

Characterization of Futures Price

- The futures price for immediate delivery at $T = 3$ is $H_{3,3} = S_3$

- The strategy turns an investment of $H_{0,3}$ into a cash flow of $\dfrac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$.

- In general, an investment of $H_{t,t+T}$ can be turned into
$$\frac{S_{t+T}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}...B_{t+T-1,t+T}}.$$

- Hence $H_{t,t+T}$ must be the value of this pay-off.

Characterization of Forward Price

[CIR proposition 2]

Consider an asset with a price $\widetilde{S}_{t+T}$ at time $t+T$. The forward price of the asset, $F_{t,t+T}$, is the time t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_{t+T}}{B_{t,t+T}}$$

at time $t+T$.

Characterization of Forward Price

- Invest $F_{t,t+T}$ in a T-maturity risk-free bond at time t at the long bond price, $B_{t,t+T}$.

- Take out $1/B_{t,t+T}$ long forward contracts to buy the asset.

- Time $t+T$, the payoff of the risk-free bond investment is $F_{t,t+T}/B_{t,t+T}$,

- Forward contract payoff is
$$\left(S_{t+T} - F_{t,t+T}\right)/B_{t,t+T}.$$

- At time $t+T$, the combined position is $S_{t+T}/B_{t,t+T}$.

- So $F_{t,t+T}$ is the time t value of $\widetilde{S}_{t+T}/B_{t,t+T}$.

Pricing under Rational Expectations

S_t is the value of x_{t+3}:

$$S_t = B_{t,t+1}E_t\{\phi_{t,t+1}B_{t+1,t+2}E_{t+1}[\phi_{t+1,t+2}$$
$$B_{t+2,t+3}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\},$$

where $\phi_{\tau,\tau+1}$, are the period-to-period pricing kernels.

$$S_t = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}$$
$$B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}x_{t+3})]\},$$

The futures price $H_{t,t+3}$ is the time-t value of
$$\frac{x_{t+3}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}}$$

$$S_t = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}$$
$$B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}\frac{x_{t+3}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}})]\}$$

$$H_{t,t+3} = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\}$$

A Great Result

$$H_{t,t+3} = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\}$$

or,

$$
\begin{aligned}
H_{t,t+3} &= E_t^Q\{E_{t+1}^Q[E_{t+2}^Q(x_{t+3})]\} \\
&= E_t^Q(x_{t+3})
\end{aligned}
$$

Futures Price: Interpretation

- Under risk neutrality, futures = expected payoff

- Futures price has 'martingale property' under Q (risk-neutral) measure

- Compounding effect offsets stochastic discounting

- Under risk neutrality, forward $\neq$ expected payoff

The Forward Price

the forward price of x_{t+3} is the spot price of an asset paying $x_{t+3}/B_{t,t+3}$ at time $t+3$.

$$F_{t,t+3} = E_t^Q \left(\frac{B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3} x_{t+3}}{B_{t,t+3}} \right).$$

$$F_{t,t+T} = E_t^Q \left[\prod_{\tau=1}^{T} \frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}} x_{t+T} \right].$$

$$b_{t,t+T} = \prod_{\tau=1}^{T-1} \frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}}.$$

$$E_t^Q(b_{t,t+T}) = 1$$

The Forward Price

$$F_{t,t+T} = E_t^Q(b_{t,t+T} x_{t+T}).$$

$$F_{t,t+T} = E_t^Q(b_{t,t+T}) E_t^Q(x_{t+T}) + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

$$F_{t,t+T} = H_{t,t+T} + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

The Forward Price: Interpretation

- Forward price = Futures price + covariance term

- Covariance may be positive for many assets

- Covariance is positive for bonds

- For bonds, Forward price > Futures price

An Example: Lognormal Variables

Assume:

- x_{t+T} is lognormal

- $\phi^*_{t,t+T} = \phi_{t,t+1}\phi_{t+1,t+2}\phi_{t+2,t+3}\ldots$ is lognormal

- $b_{t,t+T}$ is lognormal

If x, y are lognormal with

$$\mu_x, \sigma_x$$

$$\mu_y, \sigma_y$$

$$\sigma_{x,y}$$

Then

$$E(xy) = E(x)E(y)e^{\sigma_{x,y}}$$

Futures price:

$$H_{t,t+T} = E(x\phi)$$

$$= E(x)E(\phi)e^{\sigma_{x,\phi}}$$

$$= E(x)e^{\sigma_{x,\phi}}$$

$$= e^{\mu_x + \frac{1}{2}\sigma_x^2}e^{\sigma_{x,\phi}}$$

$$= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,\phi}}$$

Forward price

$$F_{t,t+T} = E(xb\phi)$$

$$= E(x)E(b\phi)e^{cov(\ln x, \ln b\phi)}$$

$$= E(x)e^{cov(\ln x, \ln b) + cov(\ln x, \ln \phi)}$$

$$= e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x,b} + \sigma_{x,\phi}}$$

$$= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,b} + \sigma_{x,\phi}}$$

The Forward and Futures Prices: Lognormal Variables

Then:

- Futures price:

$$H_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x\phi}}.$$

- Forward price

$$F_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x\phi} + \sigma_{xb}}.$$

The Forward-Futures Bias: Lognormal Variables

-

$$F_{t,t+T} = H_{t,t+T} \; e^{\sigma_{xb}}$$

-

$$\sigma_{xb} = \sigma_x \sigma_b \rho_{xb}$$

- For Bonds

$$\rho_{xb} > 0$$

- σ is non-annualised

- Forward-futures bias increases with maturity of futures

The Forward-Futures Bias: Further Results

- Interest rates: inversely related to bond prices

- Section 6.5: define futures rate by:

$$H_{t,t+T} = e^{-h_{t,t+T}}.$$

- Define forward rate by:

$$F_{t,t+T} = e^{-f_{t,t+T}}.$$

-

$$\ln\left(\frac{F_{t,t+T}}{H_{t,t+T}}\right) = h_{t,t+T} - f_{t,t+T} = \sigma_{xb},$$

- Futures rate $>$ Forward rate

- Important for interest-rate modelling (see ch 7)

The Forward-Futures Bias: Further Results

- For contingent claims (options) bias is magnified

- On LIFFE, Sydney, options traded on futures basis

- See section 6.6

The Forward-Futures Bias: Conclusions

- Forward-Futures bias depends on covariance of asset price with the bond roll-up factor

- Bias is more significant for bond futures

- Forward price $>$ Futures price

- Interest rates: Forward rate $<$ Futures rate

- Bias increases with maturity of futures contract

- Bias is magnified in the case of futures on options