

Foundations of Finance Theory: Course Notes

Richard C. Stapleton¹

¹Manchester Business School

1 Mean-variance portfolio analysis and the CAPM

Portfolio Demand and the CAPM

Assume

- $j = 1, 2, \dots, J$ firms in the economy, with N_j shares outstanding. 100% equity. Cash flow, x_j at time $t = 1$
- $i = 1, 2, \dots, I$ investors with wealth $w_{0,i}$ at $t = 0$ and utility

$$u_i = u_i[E(w_i), \text{var}(w_i)],$$

where $w_i \equiv w_{1,i}$ is wealth at time $t = 1$

- Investors have homogeneous expectations
- Investors can borrow or lend at risk-free rate of interest r_f
- No taxes, transactions costs

Definitions

Let:

- $S_j = s_j N_j$ be market capitalisation of j , s_j is price of 1 share
- $n_{i,j}$ shares bought by investor i in firm j
-

$$\alpha_{i,j} = \frac{n_{i,j}}{N_j}$$

- B_i is bond investment of investor i
- $\sigma_{j,j} = \text{var}(x_j)$
- $\sigma_{j,k} = \text{covar}(x_j, x_k)$

Hence

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f) \quad (1)$$

where

$$B_i = w_{0,i} - \sum_j \alpha_{i,j} S_j$$

Portfolio Demand

Investor i chooses shares $n_{i,j}$ to maximise utility:

$$\max u_i = u_i[E(w_i), \text{var}(w_i)] \quad (2)$$

where

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f)$$

subject to

$$B_i + \sum_j \alpha_{i,j} S_j = w_{0,i}$$

Step 1: Compute Portfolio Mean and Variance

Mean:

$$E(w_i) = w_{0,i}(1+r_f) + \sum_j \alpha_{i,j}[E(x_j) - S_j(1+r_f)] \quad (3)$$

Variance:

$$\begin{aligned} var(w_i) &= var\left(\sum_j \alpha_{i,j}x_j\right) \\ &= \alpha_{i,1}^2\sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + \dots \\ &\quad + 2\alpha_{i,1}\alpha_{i,j}\sigma_{1,j} + \alpha_{i,2}^2\sigma_{2,2} + \dots \\ &\quad + \alpha_{i,j}^2\sigma_{j,j} \end{aligned} \quad (4)$$

Hence

$$\frac{\partial E(w_i)}{\partial \alpha_{i,j}} = E(x_j) - S_j(1+r_f) \quad (5)$$

$$\begin{aligned} \frac{\partial var(w_i)}{\partial \alpha_{i,j}} &= 2\alpha_{i,1}\sigma_{j,1} + 2\alpha_{i,2}\sigma_{j,2} + \dots + 2\alpha_{i,j}\sigma_{j,j} \\ &\quad + \dots \end{aligned} \quad (6)$$

Step 2: First Order Conditions

Hence

$$\begin{aligned} \frac{\partial u_i}{\partial \alpha_{i,j}} &= \frac{\partial u_i}{\partial E(w_i)} \frac{\partial E(w_i)}{\partial \alpha_{i,j}} \\ &+ \frac{\partial u_i}{\partial var(w_i)} \frac{\partial var(w_i)}{\partial \alpha_{i,j}} = 0, \forall j \end{aligned}$$

or

$$\frac{-\frac{\partial u_i}{\partial E(w_i)} \frac{\partial E(w_i)}{\partial \alpha_{i,j}}}{\frac{\partial u_i}{\partial var(w_i)} \frac{\partial var(w_i)}{\partial \alpha_{i,j}}} = \frac{\partial var(w_i)}{\partial \alpha_{i,j}}, \forall j$$

Substituting (5) and (6)

$$\begin{aligned} \lambda_i [E(x_j) - S_j(1 + r_f)] &= \alpha_{i,1} \sigma_{j,1} + \alpha_{i,2} \sigma_{j,2} + \dots \\ &+ \alpha_{i,j} \sigma_{j,j} + \dots, \forall j \end{aligned}$$

where

$$\lambda_i = \frac{-\frac{\partial u_i}{\partial E w_i}}{2 \frac{\partial u_i}{\partial var(w_i)}}$$

Step 3: Optimal Stock Proportions

This set of simultaneous equations can be written in matrix form:

$$\lambda_i [E(x) - S(1 + r_f)] = A\alpha_i$$

where

$$E(x) = \begin{pmatrix} E(x_1) \\ E(x_2) \\ \cdot \\ \cdot \\ E(x_J) \end{pmatrix}, S = \begin{pmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ S_J \end{pmatrix}, \alpha_i = \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ \cdot \\ \cdot \\ \alpha_{i,J} \end{pmatrix}$$

and

$$A = \begin{pmatrix} \sigma_{1,1}, \sigma_{1,2}, \dots, \sigma_{1,J} \\ \sigma_{2,1}, \sigma_{2,2}, \dots, \sigma_{2,J} \\ \sigma_{3,1}, \sigma_{3,2}, \dots, \sigma_{3,J} \\ \cdot \quad \cdot \quad \cdot \\ \sigma_{J,1}, \sigma_{J,2}, \dots, \sigma_{J,J} \end{pmatrix}$$

The solution is

$$\alpha_i = \lambda_i A^{-1} [E(x) - S(1 + r_f)] \quad (7)$$

(7) proves the mutual fund *Separation Theorem*

Step 4: Market Equilibrium In equilibrium all stocks must be held, or

$$\sum_i \alpha_{i,j} = 1, \forall j$$

Summing (7) over i

$$\sum_i \alpha_i = \sum_i \lambda_i A^{-1} [E(x) - S(1 + r_f)] = \bar{\mathbf{1}},$$

where

$$\bar{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

Solving for S

$$S = \frac{1}{1 + r_f} \left(E(x) - \frac{A\bar{\mathbf{1}}}{\lambda} \right), \quad (8)$$

where

$$A\bar{\mathbf{1}} = \begin{pmatrix} \sigma_{1,1} + \sigma_{1,2} + \sigma_{1,3} + \dots + \sigma_{1,J} \\ \sigma_{2,1} + \sigma_{2,2} + \sigma_{2,3} + \dots + \sigma_{2,J} \\ \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\ \sigma_{J,1} + \sigma_{J,2} + \sigma_{J,3} + \dots + \sigma_{J,J} \end{pmatrix}$$

Step 5: The CAPM

For stock 1:

$$\begin{aligned}
 \sigma_{1,1} + \sigma_{1,2} + \sigma_{1,3} + \dots + \sigma_{1,J} &= \text{cov}(x_1, x_1) \\
 &+ \text{cov}(x_1, x_2) + \dots + \text{cov}(x_1, x_J) \\
 &= \text{cov}\left(x_1, \sum_{j=1}^J x_j\right) \\
 &= \text{cov}(x_1, x_m),
 \end{aligned}$$

where

$$x_m = \sum_{j=1}^J x_j$$

is the market cash flow.

$$S_j = \frac{1}{1 + r_f} \left[E x_j - \frac{1}{\lambda} \text{cov}(x_j, x_m) \right], \quad (9)$$

Step 6: The CAPM: Expected Return

Divide (9) by S_j ,

$$1 = \frac{1}{1 + r_f} \left[\frac{E(x_j)}{S_j} - \frac{1}{\lambda} \text{cov} \left(\frac{x_j}{S_j}, x_m \right) \right],$$

Define the return:

$$r_j = \frac{x_j - S_j}{S_j},$$

$$1 = \frac{1}{1 + r_f} \left[E(r_j) + 1 - \frac{S_m}{\lambda} \text{cov} \left(\frac{x_j}{S_j}, \frac{x_m}{S_m} \right) \right],$$

$$E(r_j) = r_f + \frac{S_m \text{var}(r_m) \text{cov}(r_j, r_m)}{\lambda \text{var}(r_m)},$$

$$E(r_j) = r_f + \lambda' \beta_j.$$

Step 7: The CAPM: Closed Form Solution

CAPM holds also for portfolios of shares

If CAPM holds for the market portfolio:

$$E(r_m) = r_f + \lambda' \beta_m.$$

But, $\beta_m = 1$, hence

$$E(r_m) = r_f + \lambda'.$$

The market risk premium:

$$E(r_m) - r_f = \lambda'.$$

and hence

$$E(r_j) = r_f + [E(r_m) - r_f] \beta_j.$$

Optimal Portfolios: Numerical Example

Mean-variance portfolio analysis and the CAPM

Data: Stocks

j	$E(x_j)$	S_j	σ_j	N_j
1	9	7.5	1.5	150
2	38	34	2	400

$$\rho_{1,2} = 0.5$$

Data: Investors

i	λ_i	$w_{0,i}$
1	0.3	10

$$r_f = 0.05$$

Optimal portfolio

Step 1: Compute portfolio mean and variance

Mean

$$E(x_j) - S_j(1 + r_f)$$

$$j = 1 : 9 - 7.5(1.05) = 1.125$$

$$j = 2 : 38 - 34(1.05) = 2.3$$

$$E(w_i) = 10(1.05) + \alpha_{i,1}(1.125) + \alpha_{i,2}(2.3)$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,1}} = 1.125$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,2}} = 2.3$$

Variance

$$\text{var}(w_i) = \alpha_{i,1}^2 \sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + \alpha_{i,2}^2 \sigma_{2,2}$$

$$\sigma_{1,2} = \sigma_1 \sigma_2 \rho_{1,2} = 1.5(2)(0.5) = 1.5$$

$$\sigma_{1,1} = \sigma_1^2 = 1.5^2 = 2.25$$

$$\sigma_{2,2} = \sigma_2^2 = 2^2 = 4$$

$$\text{var}(w_i) = \alpha_{i,1}^2 2.25 + 2\alpha_{i,1}\alpha_{i,2}1.5 + \alpha_{i,2}^2 4$$

$$\frac{\partial \text{var}(w_i)}{\partial \alpha_{i,1}} = 2\alpha_{i,1}2.25 + 2\alpha_{i,2}1.5$$

$$\frac{\partial \text{var}(w_i)}{\partial \alpha_{i,2}} = 2\alpha_{i,1}1.5 + 2\alpha_{i,2}4$$

Step 2: First order conditions

$$\lambda_i [E(x_j) - S_j(1+r_f)] = \sigma_{j,1}\alpha_{i,1} + \sigma_{j,2}\alpha_{i,2}$$

$$j = 1 : 0.3(1.125) = 2.25\alpha_{i,1} + 1.5\alpha_{i,2}$$

$$j = 2 : 0.3(2.3) = 1.5\alpha_{i,1} + 4\alpha_{i,2}$$

Step 3: Matrix form

$$\lambda_i [E(x) - S(1 + r_f)] = A\alpha_i$$

$$A = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix}$$

$$0.3 \begin{pmatrix} 1.125 \\ 2.3 \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix} \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}$$

$$\begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix} \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}$$

Hence

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 2.25 & 1.5 \\ 1.5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 0.5926 & -0.222 \\ -0.222 & 0.333 \end{pmatrix} \begin{pmatrix} 0.3375 \\ 0.69 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix} = \begin{pmatrix} 0.0467 \\ 0.155 \end{pmatrix}$$

Optimal Share allocation

$$n_{i,1} = \alpha_{i,1}(N_1)$$

$$n_{i,1} = 0.0467(150) = 7$$

$$n_{i,2} = \alpha_{i,2}(N_2)$$

$$n_{i,2} = 0.155(400) = 62$$

Stock price

$$s_1 = 7.5/150 = 0.05$$

$$s_2 = 34/400 = 0.085$$

Investment:

stocks

$$n_1 s_1 + n_2 s_2$$

$$7(0.05) + 62(0.085) = 5.62$$

bonds

$$w - 5.62 = 4.38$$

total 10.00

2 Asset pricing: a complete markets model

Financial Theory

Assets:

Value of Cash Flows

CAPM

Options:

Black-Scholes

Multi-period

Models

Forward / Futures

Prices: Assets/Options

A General Model

Initial setup and key assumptions

1. Assume a single period from t to $t + T$.
2. Assume forward parity holds.
3. Assume that there are a finite number of states of the world at time $t + T$

A state-contingent claim on state i is defined as a security which pays \$1 if and only if state i occurs.

4. Assume markets are complete.
5. Assume homogeneous expectations
6. Assume that the price of a portfolio or package of contingent claims is equal to the sum of the prices of the individual state-contingent claims.

1: Single Period Model

0 ————— 1

t ————— $t + T$

- Nothing happens between t and $t + T$
- Dividends are paid either at t or at $t + T$
- No trading between t and $t + T$

2: Forward Parity

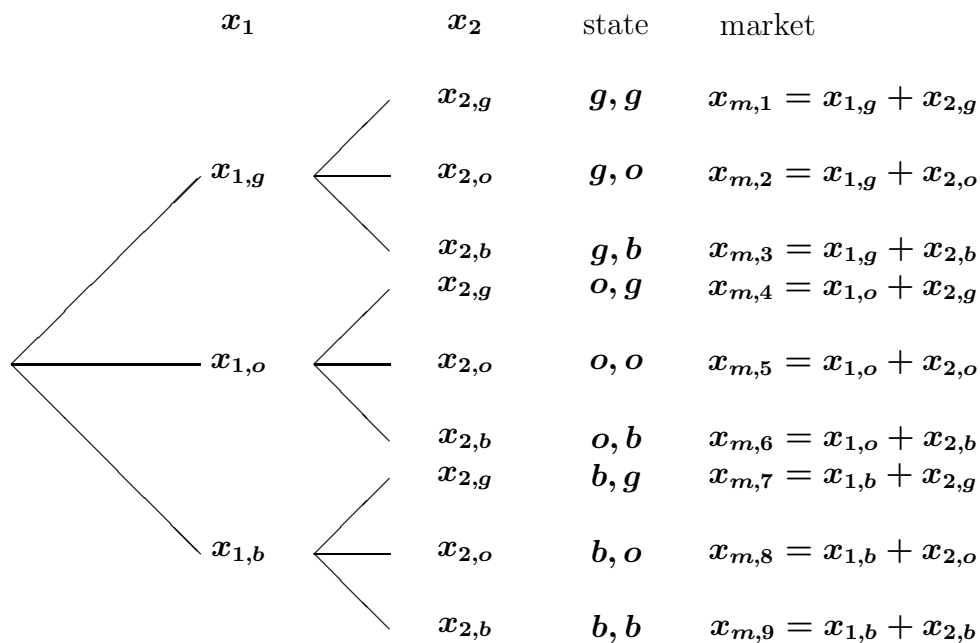
Assume no dividend paid between t and $t + T$

Synthetic forward strategy:

1. Borrow S_t (price of stock)
2. Buy stock
3. Repay $S_t(1 + r)$ at $t + T$

- Net cash flow at t is zero.
- Net cash at $t + T$ is $S_{t+T} - S_t(1 + r)$
- $S_t(1 + r)$ must be forward price
- $\frac{1}{(1+r)} = B_{t,t+T}$
- $F_{t,t+T} = \frac{S_t}{B_{t,t+T}}$

3: Finite State Space

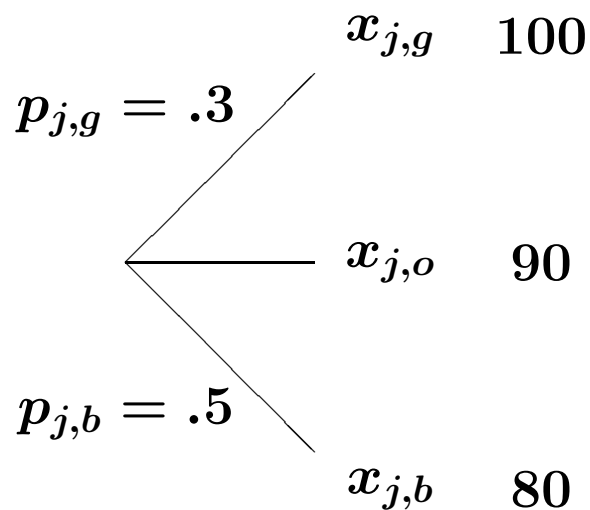


4: Complete Market

Buy (or construct) state contingent claims

- Buy an option paying \$1 if $x \geq k_1$
- Sell an option paying \$1 if $x \geq k_2$
- Portfolio pays \$1 if $k_1 < x < k_2$

5: Homogeneous Expectations



Firm j

- Investors agree on cash flows, x_j
- Investors agree on probabilities, p_j

6: Asset Forward Price

Law of One price

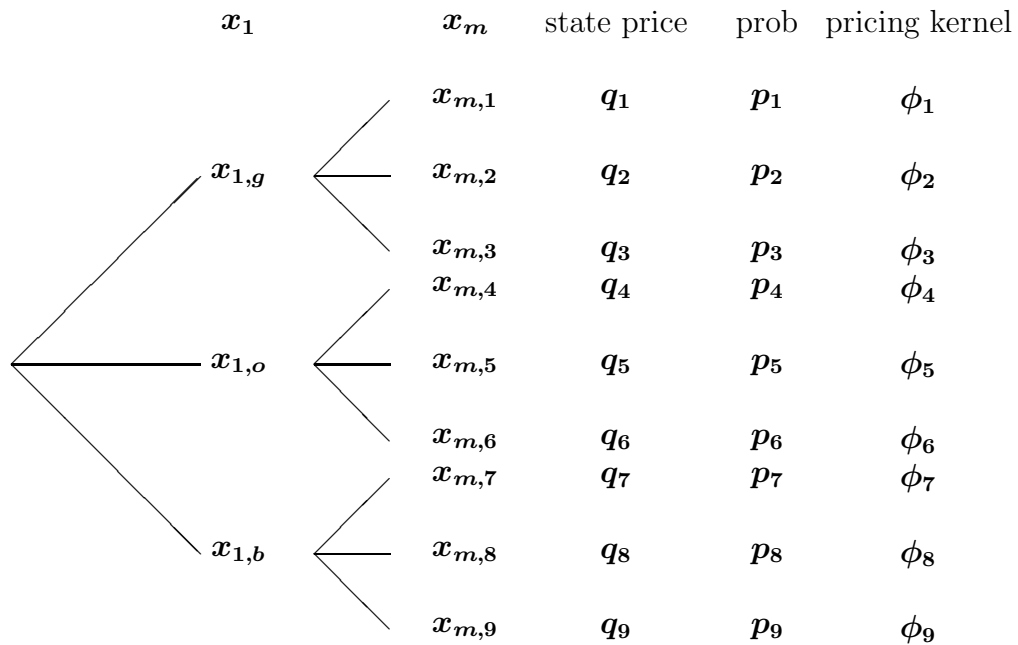
$$F_{j,t,t+T} = \sum_i (x_{j,t+T,i} q_i)$$

$$F_j = \sum_i (q_i x_{j,i})$$

Properties of the state price

1. The state price, q_i , is always greater than zero.
2. The state prices sum to 1, i.e. $\sum_i q_i = 1$.

State Price, Pricing Kernel



$$\begin{aligned}
 F_1 &= \mathbf{x}_{1,g}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\
 &+ \mathbf{x}_{1,o}(\mathbf{q}_4 + \mathbf{q}_5 + \mathbf{q}_6) \\
 &+ \mathbf{x}_{1,b}(\mathbf{q}_7 + \mathbf{q}_8 + \mathbf{q}_9)
 \end{aligned}$$

The Pricing Kernel, ϕ_i

$$\phi_i = \frac{q_i}{p_i},$$

The properties of ϕ_i are as follows:

1. Since $p_i > 0$ and $q_i > 0$, this means the pricing kernel ϕ_i is a positive function.
2. $E(\phi) = 1$.

Normality of x_m : The CAPM

$$F_j = \sum_i (x_{j,i} q_i) = \sum_i p_i (\phi_i x_{j,i}) = E(\phi x_j)$$

$$F_j = E(x_j \phi) = E(\phi) E(x_j) + \text{cov}(\phi, x_j)$$

$$F_j = E(x_j) + \text{cov}(\phi, x_j).$$

$$F_j = E(x_j) + \text{cov}[\phi(x_m), x_j].$$

Covariance and joint probability

Covariance definition

$$\text{cov}(x, y) = E\{[x - E(x)][y - E(y)]\}.$$

$$E(x) = p(x_1)x_1 + p(x_2)x_2 + \dots$$

$$E(x) = \sum_i p(x_i)x_i$$

$$E(y) = p(y_1)y_1 + p(y_2)y_2 + \dots$$

$$E(y) = \sum_i p(y_i)y_i$$

Let $p(x_i, y_j)$ be the joint probability of x_i and y_j occurring, then

$$\begin{aligned} \text{cov}(x, y) &= p(x_1, y_1)[x_1 - E(x)][y_1 - E(y)] \\ &+ p(x_1, y_2)[x_1 - E(x)][y_2 - E(y)] \\ &+ \dots \\ &+ p(x_i, y_j)[x_i - E(x)][y_j - E(y)] \end{aligned}$$

Stein's Lemma

If x, y are joint normally distributed

$$\text{cov}(g(x), y) = E[g'(x)]\text{cov}(x, y)$$

Hence

$$\text{cov}(x_j, \phi(x_m)) = E[\phi'(x_m)]\text{cov}(x_j, x_m)$$

Stein's Lemma

Examples:

1.

$$\text{cov}[g(x), y] = E[g'(x)]\text{cov}(x, y)$$

let

$$g(x) = a + bx$$

$$g'(x) = b$$

$$\text{cov}[g(x), y] = b \text{cov}(x, y)$$

2. let

$$g(x) = a + bx + cx^2$$

$$g'(x) = b + 2cx$$

$$\text{cov}[g(x), y] = [b + 2cE(x)]\text{cov}(x, y)$$

3. let

$$g(x) = e^x$$

$$g'(x) = e^x$$

$$\text{cov}[g(x), y] = E(e^x)\text{cov}(x, y)$$

CAPM: Derivation: Forward-Cash Flow Version

$$\begin{aligned}
 F_j &= E(x_j) E(\phi(x_m)) + \text{COV}(x_j, \phi(x_m)) \cdot \\
 F_j &= E(x_j) E(\phi(x_m)) - \kappa \text{COV}(x_j, x_m) \cdot \\
 &= E(x_j) - \kappa \text{COV}(x_j, x_m),
 \end{aligned}$$

$$\kappa = -E[\phi'(x_m)].$$

Since

$$F_m = E(x_m) - \kappa \text{COV}(x_m, x_m)$$

$$\kappa = \frac{E(x_m) - F_m}{\text{var}(x_m)}$$

and

$$F_j = E(x_j) - \frac{E(x_m) - F_m}{\text{var}(x_m)} \text{COV}(x_j, x_m)$$

CAPM: Applications

1. Cost of capital

- For firm
- For capital project

2. Valuation of non-listed equity

3. Mergers and acquisitions

For applications see Brealey and Myers, Copeland and Weston

APT Arbitrage Pricing Theory

We assume there are k factors and for factor f_k the factor loading is β_{jk} .

$$x_j = a_j + \sum_{k=1}^K \beta_{jk} f_k + \varepsilon_j$$

where ε_j is independent of f_k .

$$\text{cov}(x_j, \phi(x_m)) = \sum_{k=1}^K \beta_{jk} \text{cov}(f_k, \phi(x_m)) + \text{cov}(\varepsilon_j, \phi(x_m))$$

1. $\varepsilon_j = 0$

2. $\text{cov}(\varepsilon_j, \phi(x_m)) = 0$

If $\varepsilon_j = 0$ or $\text{cov}(\varepsilon_j, \phi(x_m)) = 0$, then

$$\begin{aligned} \text{cov}(x_j, \phi(x_m)) &= \sum_{k=1}^K \beta_{jk} \text{cov}(f_k, \phi(x_m)) \\ E_j &= E(x_j) + \beta_{j1} \text{cov}(f_1, \phi(x_m)) \\ &\quad + \beta_{j2} \text{cov}(f_2, \phi(x_m)) + \dots \end{aligned}$$

The Pricing Kernel: An Equilibrium Model

- Assume a 'representative agent' economy
- Let $w_{t+T,i}$ be the wealth of the investor in state i at time $t + T$.
- Assume that the investor is endowed with investible wealth w_t at time t
- The investor can purchase state-contingent claims.
- Choose a set of state-contingent claims paying $w_{t+T,i}$, given a budget allocation of cash, w_t .

We make the following additional assumptions:

1. The investor maximises the expected value of a utility function $u(w_{t+T})$. Hence the investors problem is:

$$\max E [u (w_{t+T})] = \sum_i p_i u(w_{t+T,i})$$

subject to

$$\sum_i w_{t+T,i} q_i B_{t,t+T} = w_t \quad (10)$$

2. The utility function has the properties $u'(w_{t+T}) > 0$ (non-satiation) and $u''(w_{t+T}) < 0$ (risk aversion).

A two-state example:

$$\max p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2})$$

subject to

$$[w_{t+T,1}q_1 + w_{t+T,2}q_2]B_{t,t+T} = w_t$$

Lagrangian:

$$L = p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2}) \\ + \lambda \{w_t B_{t,t+T}^{-1} - w_{t+T,1}q_1 - w_{t+T,2}q_2\}$$

FOC for a maximum:

$$\frac{\partial L}{\partial w_{t+T,1}} = p_1 u'(w_{t+T,1}) - q_1 \lambda = 0.$$

$$\frac{\partial L}{\partial w_{t+T,2}} = p_2 u'(w_{t+T,2}) - q_2 \lambda = 0.$$

Summing over the states i

$$p_1 u'(w_{t+T,1}) + p_2 u'(w_{t+T,2}) = \lambda(q_1 + q_2)$$

or

$$E[u'(w_{t+T})] = \lambda,$$

since $q_1 + q_2 = 1$.

Substituting for λ

$$\frac{p_1 u'(w_{t+T,1})}{E[u'(w_{t+T})]} = q_1,$$

$$\frac{p_2 u'(w_{t+T,2})}{E[u'(w_{t+T})]} = q_2,$$

and hence the first order conditions are satisfied if

$$\frac{u'(w_{t+T,i})}{E[u'(w_{t+T})]} = \frac{q_i}{p_i} = \phi_i, \quad \forall i.$$

$$\phi_i = \frac{u'(x_{m,i})}{E[u'(x_{m,i})]} : \forall i.$$

Hence we have

$$\phi = \phi(x_m),$$

Case 1: Risk Neutrality

$$u(w_{t+T}) = a + b w_{t+T},$$

$$u'(w_{t+T}) = b,$$

$$E[u'(w_{t+T})] = b,$$

$$\phi_i(w_{t+T,i}) = \frac{u'(w_{t+T,i})}{E[u'(w_{t+T})]} = 1.$$

Forward Price Under Risk Neutrality:

In this case, the forward price is

$$\begin{aligned} F_{t,t+T} &= E(\phi x) \\ &= E(1 \cdot x) \\ &= E(F_{t+T,t+T}) \end{aligned}$$

Case 2: Utility is Quadratic

Assume utility is given by:

$$\begin{aligned}
 u(w_{t+T}) &= a + bw_{t+T} + \delta w_{t+T}^2 \\
 u'(w_{t+T}) &= b + 2\delta w_{t+T} \\
 \phi(w_{t+T}) &= \frac{u'(w_{t+T})}{E[u'(w_{t+T})]} \\
 &= \frac{b + 2\delta w_{t+T}}{b + 2\delta E(w_{t+T})} \\
 \text{cov}[\phi(w_{t+T}), x_{t+T}] &= \frac{2\delta \text{cov}(w_{t+T}, x_{t+T})}{b + 2\delta E(w_{t+T})}
 \end{aligned}$$

Then

$$F_{t,t+T} = E(x_{t+T}) + \kappa \text{cov}(w_{t+T}, x_{t+T}),$$

where

$$\kappa = \frac{2\delta}{b + 2\delta E(w_{t+T})}$$

Asset specific pricing kernel

For asset j , we can write

$$\begin{aligned} F_j &= E(\phi x_j) \\ &= E[E(\phi | x_j) x_j] \end{aligned}$$

$$F_j = E(\psi_j x_j)$$

$$\psi_j > 0. \quad E(\psi_j) = 1.$$

Example:

Figure 1.2

$$\psi_{x_{1,g}} = E(\phi | x_1 = x_{1,g})$$

$$\psi_{x_{1,g}} = p_1 \phi_1 + p_2 \phi_2 + p_3 \phi_3$$

Other Readings:

Huang C-F and R.H. Litzenberger (1988)
Foundations for Financial Economics,
North Holland, Chapters 1 and 4.

Pliska (1997) Introduction to Mathe-
matical Finance, Blackwell, Chap-
ters 1 and 2.

Cochrane (2001) Asset Pricing, Prince-
ton, chs 1-6

Copeland and Weston (1988), Finan-
cial Theory and Corporate Policy,
Addison-Wesley.

3 Option pricing and risk-neutral valuation

Options and Contingent Claims

A contingent claim on a cash flow x pays some function $g(x)$. The payoff $g(x)$ is contingent on the payoff x on the underlying asset.

A European-style call option maturing at time $t + T$, on a cash flow x pays

$$g(x) = \max[x - k, 0]$$

A European-style put option maturing at time $t + T$, on a cash flow x pays

$$g(x) = \max[k - x, 0]$$

The Normal Distribution

If $f(x)$ is normal with mean μ_x , and standard deviation σ_x :

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [x - \mu_x]^2}$$

The standard normal distribution, has mean 0 and standard deviation 1 is

$$n(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

The cumulative normal distribution function is

$$F(x) = \int_{-\infty}^x f(u) du$$

The cumulative standard normal distribution function is

$$N(y) = \int_{-\infty}^y f(u) du$$

- If $f(x)$ normal,

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

-

$$E(x > a) = \int_a^{\infty} x f(x) dx$$

- The normal distribution is the limit $n \rightarrow \infty$ of a binomial distribution with n steps

The Joint Normal Distribution

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y}\right]}$$

- If $f(x, y)$ joint normal, $f(x)$ and $f(y)$ are normal
- If $f(x, y)$ joint normal, the conditional distribution $f(x|y)$ is normal
- The regression of x on y is

$$E(x|y) = \mu_x + \frac{\rho\sigma_x}{\sigma_y}(y - \mu_y)$$

A Simple Option Pricing Model

Assume quadratic utility for the representative investor *and* joint-normal distribution

From ch 1

$$\begin{aligned}
 u(x_m) &= a + bx_m + \delta x_m^2 \\
 \phi(x_m) &= \frac{u'(x_m)}{E[u'(x_m)]} \\
 &= \frac{b + 2\delta x_m}{b + 2\delta E(x_m)}
 \end{aligned}$$

Hence in this case

$$\phi = A + Bx_m.$$

$$F[g(x)] = E[g(x)] + \text{cov}(g(x), A + Bx_m)$$

Using Stein's lemma

$$F[g(x)] = E[g(x)] + BE[g'(x)] \text{cov}(x, x_m)$$

The contingent claim is a call option where $g(x) = \max(x - k, 0)$.

$E[g'(x)] = \text{prob}(x > k)$ and hence

$$F[g(x)] = E[g(x)] + B[\text{prob}(x > k)] \text{cov}(x, x_m) \quad (11)$$

$$F_m = E[x_m] + B \text{var}(x_m)$$

$$B = \frac{F_m - E(x_m)}{\text{var}(x_m)}$$

An application: Valuation of Corporate debt

Assumptions:

1. Company assets produces cash flow x_j at time $t + T$
2. x_j and x_m are joint normal
3. Bonds outstanding B_{t+T} to be repaid at time $t + T$
4. Equity is call option receives:

$$g(x) = \max[x_j - B_{t+T}, 0]$$

at $t + T$

5. Forward value of equity is

$$F[g(x)] = E[g(x)] + B [\text{prob}(x > k)] \text{cov}(x_j, x_m)$$

6. Spot value of equity is

$$S_t = F[g(x)] B_{t,t+T}$$

7. Value of company assets is V_t

8. Value of the debt is

$$V_t - S_t$$

The Black-Scholes Model

- Purpose
 - To price options on stocks
 - To value corporate liabilities
 - To evaluate credit risk
- Assumptions
 - Lognormal asset price
 - No dividends
 - Option is European-style
- Main Features
 - Preference-free relationship
 - Mean of asset not required
 - Inputs: asset price, volatility
- Generalisations
 - Forex options
 - American-style options

Pricing of Contingent Claims

$$\begin{aligned} F_{t,t+T} [g(x_{j,t+T})] &= \sum_i q_i g(x_{j,t+T,i}) \\ &= \sum_i p_i \phi_i g(x_{j,t+T,i}) \\ &= E [g(x_{j,t+T}) \phi] \\ &= E [g(x_j) E(\phi | x_j)] \\ &= E [g(x_j) \psi(x_j)] \end{aligned}$$

Characteristics of the Pricing Kernel, ψ_j

Assume that

$$\psi_j = \alpha x_j^\beta$$

where α and β are constants.

The asset specific pricing kernel has constant elasticity.

The elasticity of the pricing kernel defined by the relationship:

$$\eta = -\frac{\partial \psi_j / \psi_j}{\partial x_j / x_j}$$

Special Case of Options on x_m

Conditions for constant elasticity

- In this case $\psi_j = \phi(x_m)$
- $\phi(x_m)$ has constant elasticity if the representative investor has power utility
- If $u(x_m) = x_m^\gamma$ then ϕ has constant elasticity

The Lognormal Distribution

If $\ln(x)$ is normal with

$$E[\ln(x)] = \mu_x, \text{var}[\ln(x)] = \sigma_x^2$$

then x is lognormally distributed

If $f(\ln x)$ is normal with mean μ_x , and standard deviation σ_x :

$$f(\ln x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x - \mu_x]^2}$$

- The lognormal distribution is the limit of a process where $\ln x$ is binomial
- Assume a stock price S_t goes up by a proportion u with probability p or down by d with probability $(1 - p)$ then S_t is log-binomial

Further Properties of Lognormal Variables

1. The expected value of a lognormal variable: Let Y be lognormal, with $y = \ln(Y)$

$$E(Y) = E(e^y) = e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

2. Also,

$$E(Y^b) = E(e^{by}) = e^{b\mu_y + \frac{1}{2}b^2\sigma_y^2}.$$

3. If x, y are lognormal, xy is lognormal.

$$\ln xy = \ln x + \ln y$$

It follows

$$E(\ln xy) = E(\ln x) + E(\ln y)$$

or

$$E(\ln xy) = \mu_x + \mu_y$$

$$\begin{aligned} \text{var}(\ln xy) &= \text{var}(\ln x) + \text{var}(\ln y) \\ &\quad + 2\text{cov}(\ln x, \ln y) \end{aligned}$$

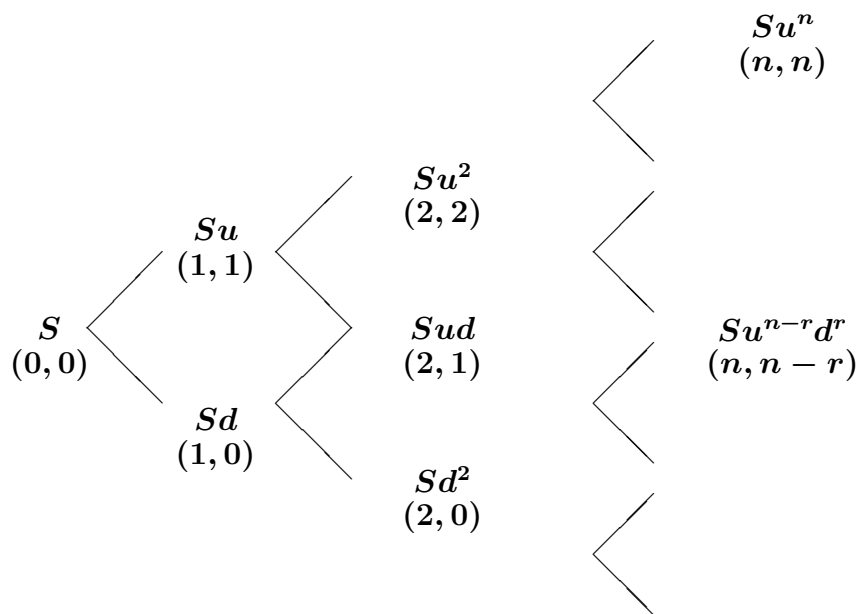
or

$$\text{var}(\ln xy) = \sigma_x^2 + \sigma_y^2 + 2\sigma_{x,y}$$

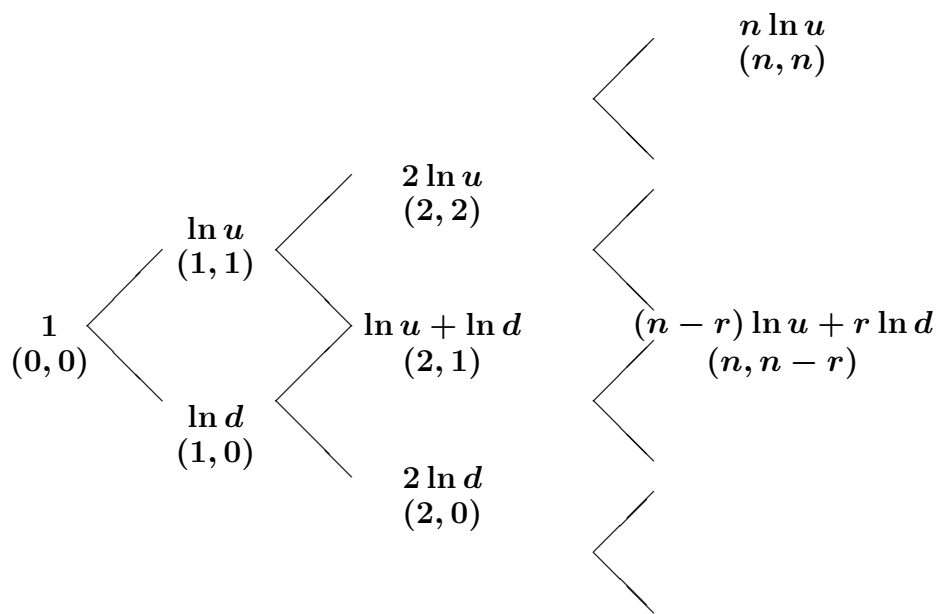
4. It follows:

$$\begin{aligned} E(xy) &= e^{E(\ln xy) + \frac{1}{2}\text{var}(\ln xy)} \\ &= e^{\mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_y^2 + \sigma_{x,y}} \end{aligned}$$

A General Binomial Process



A Log-Binomial Process



Lognormal x_j and the Black-Scholes Model

Properties of Lognormality

Pricing
Kernel

Risk-adjusted
PDF

Forward
Price of x_j

RNVR

Black-Scholes
Model

Module 1: Notation and Lognormal Properties

$$E[\ln x] = \mu_x$$

$$\text{var}[\ln x] = \sigma_x^2$$

$$f(\ln x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x - \mu_x]^2}$$

$$E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E(x_j^\beta) = e^{\beta\mu_x + \frac{1}{2}\beta^2\sigma_x^2}$$

Option payoff:

$$g(x_j) = g(e^{\ln x_j}) = h(\ln x_j)$$

Module 2: Asset Specific Pricing Kernel

$$\begin{aligned}
 F[g(x_j)] &= E[g(x_j)\phi(x_m)] \\
 &= E[g(x_j)E[\phi(x_m)|x_j]] \\
 &= E[g(x_j)\psi(x_j)]
 \end{aligned}$$

Assume that

$$\psi_j = \alpha x_j^\beta$$

where α and β are constants.

Elasticity:

$$\eta = -\frac{\partial \psi_j / \psi_j}{\partial x_j / x_j}$$

$$\eta = -\alpha \beta x_j^{\beta-1} \frac{x_j}{\alpha x_j^\beta} = -\beta$$

$E(\psi_j) = 1$ implies that

$$\alpha = e^{-\beta \mu_x - \frac{1}{2} \beta^2 \sigma_x^2}.$$

Module 3: Risk-Adjusted PDF

PDF of x_j :

$$f(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - \mu_x]^2}$$

Risk-adjusted PDF of x_j :

$$\hat{f}(\ln x_j) = f(\ln x) \psi(x_j) = \alpha x_j^\beta \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - \mu_x]^2}$$

$$\hat{f}(\ln x_j) = e^{-\beta \mu_x - \frac{1}{2} \beta^2 \sigma_x^2} x_j^\beta \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - \mu_x]^2}$$

$$\hat{f}(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - (\mu_x + \beta \sigma_x^2)]^2}$$

Module 4: Forward Asset Price of x_j

From ch 1:

$$F_j = E[x_j \psi(x_j)]$$

$$F_j = e^{\mu_x + \mu_\psi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\psi^2 + \sigma_{x\psi}}$$

since $E(\psi) = 1$

$$F_j = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x\psi}}$$

$$\ln F_j - \frac{1}{2}\sigma_x^2 = \mu_x + \sigma_{x\psi}$$

Module 5: The Lognormal RNVR

Under Risk Neutrality:

$$\begin{aligned} E[g(x_j)] &= E[h(\ln x_j)] \\ &= \int h(\ln x_j) f(\ln x_j) d \ln x_j \end{aligned}$$

$$F_j = E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E[g(x_j)] = \int h(\ln x_j) \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d \ln x_j$$

Under Risk Aversion

$$F[g(x_j)] = \int h(\ln x_j) \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d \ln x_j$$

Module 6: Option Prices: The Black-Scholes Model

$$F[g(x_j)] = E[g(x_j) \psi(x_j)]$$

$$F[g(x_j)] = \int h(\ln x_j) \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d \ln$$

The Black-Scholes Model: A Call Option

A European-style call option on x_j , with strike price k has a payoff at time $t+T$:

$$g(x_j) = \max(x_j - k, 0).$$

or

$$h(\ln x_j) = \max(e^{\ln x_j} - e^{\ln k}, 0).$$

$$F[g(x)] = \int_{-\infty}^{\infty} \max(e^{\ln x_j} - k, 0) \hat{f}(\ln x_j) d \ln x_j$$

where

$$\hat{f}(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2} [\ln x_j - (\ln F_j - \frac{1}{2}\sigma_x^2)]^2}$$

$$F[g(x)] = \int_{\ln k}^{\infty} (e^{\ln x_j} - k) \hat{f}(\ln x_j) d \ln x_j$$

$$\begin{aligned} F[g(x)] &= \int_{\ln k}^{\infty} e^{\ln x_j} \hat{f}(\ln x_j) d \ln x_j \\ &\quad - k \int_{\ln k}^{\infty} \hat{f}(\ln x_j) d \ln x_j. \end{aligned}$$

The Normal Distribution

Probability $y > a$

and

$$\int_a^{\infty} f(y) dy.$$

$$\int_a^{\infty} f(y) dy = 1 - N\left(\frac{a - \mu_y}{\sigma_y}\right) = N\left(\frac{\mu_y - a}{\sigma_y}\right).$$

The Normal Distribution

Expected value of e^y given $y > a$

$$\int_a^\infty e^y f(y) dy$$

$$\int_a^\infty e^y f(y) dy = N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

Evaluate the integrals

$y = \ln x_j$ and $a = \ln k$.

The Normal Distribution

Applying the above results and substituting the mean of $\hat{f}(\ln x_j)$, $\mu_x = \ln(F_j) - \frac{\sigma_x^2}{2}$ and $a = \ln k$, we have

$$N\left(\frac{\mu_y - a}{\sigma_y}\right) = N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k}{\sigma_x}\right)$$

and

$$\begin{aligned} & N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2} \\ &= F_j N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k + \sigma_x^2}{\sigma_x}\right) \end{aligned}$$

The Black-Scholes Model: Forward Price

$$F[g(x)] = F_j N\left(\frac{\ln \frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right) - k N\left(\frac{\ln \frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

Asset forward price F_j

The (logarithmic) variance σ_x^2 ,

The strike price of the option, k .

The Black-Scholes Model: Spot Price

$$S_t[g(x_j)] = B_{t,t+T}F_jN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right) - B_{t,t+T}kN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

or, using conventional notation:

$$S_t[g(x_j)] = B_{t,t+T}F_jN(d_1) - B_{t,t+T}kN(d_2),$$

where

$$d_1 = \frac{\ln\frac{F_j}{k} + \frac{\sigma_x^2}{2}}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x.$$

The Black-Scholes Model: Applications

1. Non-dividend paying assets:
2. Assets paying a non-stochastic dividend:
3. Assets paying a stochastic proportional dividend

The Black-Scholes Model: Non-dividend paying assets

In this case, spot-forward parity for the underlying asset means that the spot price of the asset is

$$S_t = F_j B_{t,t+T},$$

where $B_{t,t+T} = e^{-rT}$ and r is the continuously compounded interest rate.

$$S_t[g(x_j)] = S_t N(d_1) - ke^{-rT} N(d_2), \quad (12)$$

$$d_1 = \frac{\ln \frac{S_t}{k} + rT + \frac{\sigma_x^2}{2}}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x$$

The Black-Scholes Model: Assets paying a non-stochastic dividend

known dividend D_{t+T} at time $t + T$.

In this case, spot-forward parity implies

$$S_t = (F_j + D_{t+T})B_{t,t+T},$$

$$S_t[g(x_j)] = (S_t - D_{t+T}e^{-rT})N(d_1) - ke^{-rT}N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S_t - D_{t+T}e^{-rT}}{k} + \frac{\sigma_x^2}{2} + rT}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x.$$

The Black-Scholes Model: Assets paying a stochastic proportional dividend

$$D_{t+T} = \delta x_j$$

$$S_t = F_j(1 + \delta)B_{t,t+T}.$$

$$S_t[g(x_j)] = \frac{S_t}{(1 + \delta)} N(d_1) - ke^{-rT} N(d_2).$$

where

$$d_1 = \frac{\ln \frac{S_t}{(1+\delta)} + rT - \ln k + \frac{\sigma_x^2}{2}}{\sigma_x}$$

$$d_2 = d_1 - \sigma_x.$$

Spot-Forward Parity

1. Non-dividend paying assets:

$$S_t = F_j B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}}$$

2. Assets paying a non-stochastic dividend:

$$S_t = (F_j + D_{t,t+T}) B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}} - D_{t+T}$$

3. Assets paying a stochastic proportional dividend

$$S_t = F_j(1 + \delta) B_{t,t+T}.$$

$$F_j = \frac{S_t}{B_{t,t+T}} \left(\frac{1}{1 + \delta} \right)$$

The Black-Scholes Model: Extensions

- 1. Multivariate Options:**
- 2. Compound Options:**
- 3. Bermudan-style options**
- 4. American-style options**
- 5. Non-Constant Elasticity**
- 6. Non-Lognormal Asset Distributions**

4 Multi-period asset pricing

Asset prices in a Multi-period Economy

1. Chapter 5: Extension of ch 1 model to multiple periods
2. Chapter 6: Application to analysis of Forward contracts and Futures contracts

Motivation

1. Valuation of a company: more realistic model
2. Capital budgeting: Should an Investment project be accepted
3. Does the CAPM extend and apply in a multi-period world?
4. Basic problem
 - Multiple dates
 - Multiple states at each date

Asset prices in a Multi-period Economy: Introduction

- **Valuation under certainty**
- **Time-state preference approach**
- **Rational expectations approach**

Asset prices in a Multi-period Economy: Certainty Case

$$\begin{array}{cccc}
 t + 1 & t + 2 & \dots & t + n \\
 X_{j,t+1} & X_{j,t+2} & \dots & X_{j,t+n} \\
 B_{t,t+1} & B_{t,t+2} & \dots & B_{t,t+n}
 \end{array}$$

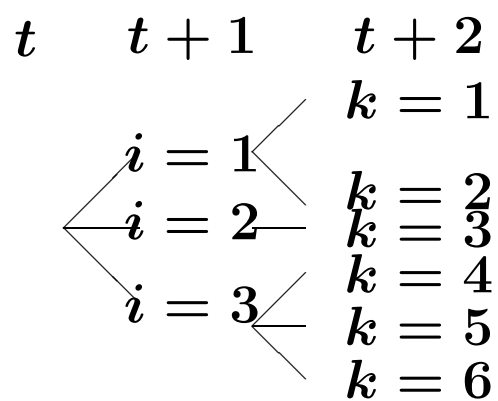
$$\begin{aligned}
 S_{j,t} &= B_{t,t+1}(X_{j,t+1} + S_{j,t+1}) \\
 S_{j,t+1} &= B_{t+1,t+2}(X_{j,t+2} + S_{j,t+2}) \\
 S_{j,t+2} &= B_{t+2,t+3}(X_{j,t+3} + S_{j,t+3}) \\
 &\dots = \dots \\
 S_{j,t+n-1} &= B_{t+n-1,t+n}X_{j,t+n}
 \end{aligned}$$

$$\begin{aligned}
 S_{j,t} &= B_{t,t+1}X_{j,t+1} + B_{t,t+1}B_{t+1,t+2}X_{j,t+2}\dots \\
 &+ B_{t,t+1}B_{t+1,t+2}\dots B_{t+n-1,t+n}X_{j,t+n}
 \end{aligned}$$

Asset prices in a Multi-period Economy: Basic Set-up

1. Markets are complete. Investors can purchase claims that pay \$1 if and only if a given state occurs at a given point in time.
2. Assume there are just two periods; period $t-t+1$ and period $t+1-t+2$.
3. Value a stock j which pays dividends $x_{j,t+1}$ at time $t+1$ and $x_{j,t+2}$ at time $t+2$.
4. There are $i = 1, 2, \dots, I$ states at time $t+1$ and there are $k = 1, 2, \dots, K$ states at time $t+2$.

Asset prices in a Multi-period Economy: Basic Set-up



Time-State Preference Approach

1. Treat each period cash flow as separate valuation
2. Define $\phi_{t,t+1} = q_i/p_i$
3. Define $\phi_{t,t+2} = q_k/p_k$
and so on.

Then applying ch 1 for each T :

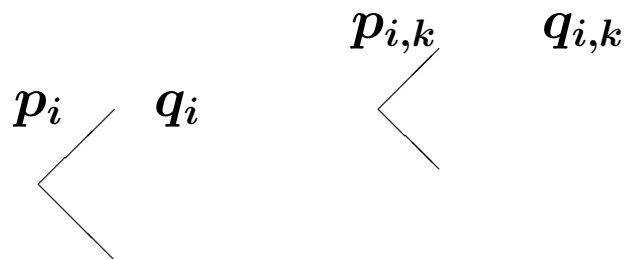
$$S_t = \sum_{T=1}^n F_{t,t+T} B_{t,t+T},$$

where

$$F_{t,t+T} = E [\phi_{t,t+T} x_{t+T}].$$

Rational Expectations Approach: Basic Set-up

Consider the following state diagram:-



p_i is the probability of state i occurring at time $t + 1$

$p_{i,k}$ is the conditional probability of state k occurring at $t + 2$

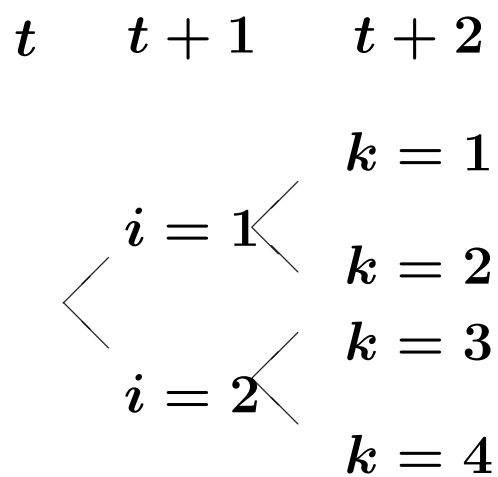
q_i is the forward price of a dollar paid *if and only if* state i occurs.

$q_{i,k}$ is the (conditional) forward price, in state i of a dollar paid *if and only if* state k occurs.

Equivalence of TSP and RE Prices: Example

Let $i = 1, 2$, $k = 1, 2, 3, 4$

Assume a state space as follows:



Equivalence of TSP and RE Prices: Example

1. Zero-coupon bond prices

$$B_{t,t+1} = 0.9$$

$$B_{t+1,t+2,1} = 0.9$$

$$B_{t+1,t+2,2} = 0.8$$

2. Forward state prices

$$q_i, i = 1 : q_1 = 4/9$$

$$q_i, i = 2 : q_2 = 5/9$$

$$q_{i,k}, i = 1, k = 1 : q_{1,1} = 4/9$$

$$q_{i,k}, i = 1, k = 2 : q_{1,2} = 5/9$$

$$q_{i,k}, i = 2, k = 3 : q_{2,3} = 1/2$$

$$q_{i,k}, i = 2, k = 4 : q_{2,4} = 1/2$$

Equivalence of TSP and RE Prices: Example

t	$t + 1$	$t + 2$
?		1
$(4/9)(0.9) \cdot (4/9)(0.9)$	0.4 -0.4	1

Note: 0.4 is spot price at date 1, in state 1

Strategy ensures \$1 in state 1 at date 2

Rational Expectations Approach

Using ch 1 method:

$$\begin{aligned} S_t &= B_{t,t+1} \sum_i p_i \phi_{t,t+1,i} S_{t+1,i} \\ &= B_{t,t+1} E_t (\phi_{t,t+1} S_{t+1}) \end{aligned}$$

This is the value of x_{t+2} at time t .

Substituting for S_{t+1} :

$$S_t = B_{t,t+1} E_t [\phi_{t,t+1} B_{t+1,t+2} E_{t+1} (\phi_{t+1,t+2} x_{t+2})]$$

The Fundamental Equation of Valuation

The value of any cash flow $x_{j,t+T}$:

$$S_{j,t} = B_{t,t+1} E_t [\phi_{t,t+1} B_{t+1,t+2} E_{t+1} (\phi_{t+1,t+2} \dots B_{t+T-1,t+T} E_{t+T-1} (\phi_{t+T-1,t+T} x_{j,t+T}))]$$

where $\phi_{\tau,\tau+1}$ is the period-by-period pricing kernel.

The Relationship Between the Pricing Kernels when Interest Rates are Non-stochastic

The cost of a claim on state k is $B_{t,t+2}q_k$.

Alternatively, cost is

$$B_{t,t+1}q_i B_{t+1,t+2}q_{i,k}.$$

Hence

$$B_{t,t+2}q_k = B_{t,t+1}q_i B_{t+1,t+2}q_{i,k}.$$

$$B_{t,t+2} \frac{q_k}{p_k} = B_{t,t+1} \frac{q_i}{p_i} B_{t+1,t+2} \frac{q_{i,k}}{p_{i,k}},$$

and hence

$$B_{t,t+2}\phi_{t,t+2} = B_{t,t+1}\phi_{t,t+1} B_{t+1,t+2}\phi_{t+1,t+2}.$$

But since in this case, $B_{t,t+2} = B_{t,t+1}B_{t+1,t+2}$, we have

$$\phi_{t,t+2} = \phi_{t,t+1} \phi_{t+1,t+2}.$$

or, in general

$$\phi_{t,t+T} = \phi_{t,t+1} \phi_{t+1,t+2} \cdots \phi_{t+T-1,t+T}.$$

Rational Expectations Approach: Basic Set-up

We now define the period-by-period pricing kernel by the relationship:

$$\phi_{t+1,t+2,i,k} = \frac{q_{i,k}}{p_{i,k}}.$$

Using this we can write:

$$\begin{aligned} S_{t+1,i} &= B_{t+1,t+2,i} \sum_k p_{i,k} \phi_{t+1,t+2,i,k} x_{t+2,k} \\ &= B_{t+1,t+2,i} E_{t+1,i} (\phi_{t+1,t+2,i} x_{t+2}) \end{aligned}$$

This is the value of x_{t+2} at time $t + 1$ in state i .

The Structure of State-Contingent Claim Prices

	Time t	Time $t + 1$	Time $t + 2$
Spot price at t of \$1 paid at $t + 2$	q_k^*		
Spot price at t of \$1 paid at $t + 1$	q_i^*		
Spot price at $t + 1$ of \$1 paid at $t + 2$		$q_{i,k}^*$	
Period – by – period valuation	$q_i^* q_{i,k}^*$	$q_{i,k}^*$	\$1
One – long – period valuation	q_k^*		\$1

$$\begin{aligned}
 q_i^* &= q_i B_{t,t+1} \\
 q_k^* &= q_k B_{t,t+2} \\
 q_{i,k}^* &= q_{i,k} B_{t+1,t+2,i}
 \end{aligned}$$

Equivalence of TSP and RE Prices

t	$t + 1$	$t + 2$
$-q_k B_{t,t+2}$		1
$-q_i B_{t,t+1} q_{i,k} B_{t+1,t+2,i}$	$+q_{i,k} B_{t+1,t+2,i}$ $-q_{i,k} B_{t+1,t+2,i}$	1

t	$t + 1$	$t + 2$
$-q_k^*$		1
$-q_i^* q_{i,k}^*$	$+q_{i,k}^*$ $-q_{i,k}^*$	1

Rational Expectations Equilibrium: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1} E_t[\phi_{t,t+1} B_{t+1,t+2} E_{t+1}(\phi_{t+1,t+2} x_{j,t+2})]$$

Alternatively, this pricing equation can be written as

$$S_{j,t} = B_{t,t+1} E_t[\phi_{t,t+1} S_{j,t+1}],$$

where

$$S_{j,t+1,i} = B_{t+1,t+2,i} E_{t+1,i}(\phi_{t+1,t+2,i} x_{j,t+2}).$$

From ch 1, the CAPM holds over $t + 1$ to $t + 2$:

$$S_{j,t+1,i} = B_{t+1,t+2,i} [E_{t+1,i}(x_{j,t+2}) - \lambda_{t+1,i} \text{cov}_{t+1,i}(x_{j,t+2}, x_{m,t+2})].$$

where $\lambda_{t+1,i}$ is the market price of risk in state i at $t + 1$

Multi-period Valuation: Joint-Normal Cash Flows

$$S_{j,t+1,i} = B_{t+1,t+2,i} [E_{t+1,i}(x_{j,t+2}) - \lambda_{t+1,i} \text{cov}_{t+1,i}(x_{j,t+2}, \mathbf{x}_{m,t+2})].$$

Now assume:

1. $B_{t+1,t+2,i}$ is non-stochastic
2. $\lambda_{t+1,i}$ is non-stochastic

These assumptions were made by Stapleton and Subrahmanyam *Econometrica* (1978)

Then it follows that:

The value, $S_{j,t+1,i}$, is normally distributed.

Also $S_{m,t+1,i}$, is normally distributed and the conditions for the period-by-period CAPM hold.

Multi-period Valuation: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1} [E_t(S_{j,t+1}) - \lambda_t \text{cov}(S_{j,t+1}, S_{m,t+1})].$$

Substituting for the time $t + 1$ prices, we can write:

$$\begin{aligned} S_{j,t} &= B_{t,t+1} B_{t+1,t+2} E_t(x_{j,t+2}) \\ &\quad - B_{t,t+2} \lambda_t \text{cov}_t [E_{t+1}(x_{j,t+2}), E_{t+1}(x_{m,t+2})] \\ &\quad - B_{t,t+2} \lambda_{t+1} \text{cov}_{t+1} [x_{j,t+2}, x_{m,t+2}] \end{aligned}$$

Multi-Period Valuation: Extensions

1. If $B_{t+1,t+2,i}$ is stochastic: Intertemporal CAPM (Merton, Long)
2. Equilibrium model to determine $\phi_{t,t+1}$, ch 5.8
3. Consumption CAPM, see Cochrane
4. RE approach, see Pliska
5. Applications: Futures v Forward prices

5 Forward and futures prices

Forward Contracts and Futures Contracts: Motivation

- Futures contracts are the way most contracts are traded on public exchanges
- In the case of interest-rate related securities the difference between forward and futures prices is important
- Determine futures prices of options (as traded on LIFFE)
- Futures prices are expected values (under risk-neutral measure)

Forward Contracts and Futures Contracts

- A *forward contract* is an agreement made at a point in time t to purchase or sell an asset at a later date $t + T$.
- A *futures contract* is similar to a forward contract but is marked to market on a daily basis as time progresses from t to $t + T$.

Futures Contracts: Payoffs

A long futures contract made at time t , with maturity T , to buy an asset at a price $H_{t,t+T}$ has a payoff $[H_{t+\tau,t+T} - H_{t+\tau-1,t+T}]$ at time $t + \tau$.

Hence the payoff at time $t+1$ is $H_{t+1,t+T} - H_{t,t+T}$

A short futures contract made at time t , with maturity T , to sell an asset at a price $H_{t,t+T}$ has a payoff $[-H_{t+\tau,t+T} + H_{t+\tau-1,t+T}]$ at time $t + \tau$.

Hence the payoff at time $t+1$ is $-H_{t+1,t+T} + H_{t,t+T}$

Forward Contracts: Payoffs

The payoff on a long forward contract is $[S_{t+T} - F_{t,t+T}]$ at time $t + T$,

The payoff on a short forward contract is $[-S_{t+T} + F_{t,t+T}]$ at time $t + T$

Forward and Futures: Payoffs

t	$t + 1$	$t + 2$	$t + 3$	\dots	$t + T$
Futures contract	$H_{t+1,t+T}$ $-H_{t,t+T}$	$H_{t+2,t+T}$ $-H_{t+1,t+T}$	$H_{t+3,t+T}$ $-H_{t+2,t+T}$	\dots	$H_{t+T,t+T}$ $-H_{t+T-1,t+T}$
Forward contract					S_{t+T} $-F_{t,t+T}$

Long Futures and Long Forward

- Futures is marked-to-market daily
- Forward pays off at end of contract

Characterization of Futures Price

[CIR (1981) Proposition 1]

Consider an asset with a price \widetilde{S}_{t+T} at time $t + T$. The futures price of the asset, $H_{t,t+T}$, is the time t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_{t+T}}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{t+T-1,t+T}}$$

at time $t + T$.

Cox J.C., J. E. Ingersoll Jr and S. A. Ross (1981) The relationship between forward prices and futures prices, *Journal of Financial Economics*, 9, pp.321-346.

A Three-year Example

Strategy

- Time 0: invest $H_{0,3}$ overnight
- Time 0: $\frac{1}{B_{0,1}}$ long futures
- Time 1: re-invest proceeds overnight
- Time 1: $\frac{1}{B_{0,1}B_{1,2}}$ long futures
- Time 2: re-invest proceeds overnight
- Time 2: $\frac{1}{B_{0,1}B_{1,2}B_{2,3}}$ long futures

Characterization of Futures Price

Time	Profits from futures	Gain from investment	Net position
0	—	—	$H_{0,3}$
1	$\frac{1}{B_{0,1}} [H_{1,3} - H_{0,3}]$	$\frac{1}{B_{0,1}} [H_{0,3}]$	$\frac{1}{B_{0,1}} [H_{1,3}]$
2	$\frac{1}{B_{0,1}B_{1,2}} [H_{2,3} - H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}} [H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}} [H_{2,3}]$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}} [H_{3,3} - H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}} [H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}} [H_{3,3}]$

Characterization of Futures Price

- The futures price for immediate delivery at $T = 3$ is $H_{3,3} = S_3$
- The strategy turns an investment of $H_{0,3}$ into a cash flow of $\frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$.
- In general, an investment of $H_{t,t+T}$ can be turned into

$$\frac{S_{t+T}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}\dots B_{t+T-1,t+T}}.$$
- Hence $H_{t,t+T}$ must be the value of this payoff.

Characterization of Forward Price

[CIR proposition 2]

Consider an asset with a price \widetilde{S}_{t+T} at time $t + T$. The forward price of the asset, $F_{t,t+T}$, is the time t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_{t+T}}{B_{t,t+T}}$$

at time $t + T$.

Characterization of Forward Price

- Invest $F_{t,t+T}$ in a T -maturity risk-free bond at time t at the long bond price, $B_{t,t+T}$.
- Take out $1/B_{t,t+T}$ long forward contracts to buy the asset.
- Time $t + T$, the payoff of the risk-free bond investment is $F_{t,t+T}/B_{t,t+T}$,
- Forward contract payoff is $(S_{t+T} - F_{t,t+T}) / B_{t,t+T}$.
- At time $t + T$, the combined position is $S_{t+T} / B_{t,t+T}$.
- So $F_{t,t+T}$ is the time t value of $\tilde{S}_{t+T} / B_{t,t+T}$.

Pricing under Rational Expectations

S_t is the value of x_{t+3} :

$$S_t = B_{t,t+1} E_t \{ \phi_{t,t+1} B_{t+1,t+2} E_{t+1} [\phi_{t+1,t+2} B_{t+2,t+3} E_{t+2} (\phi_{t+2,t+3} x_{t+3})] \},$$

where $\phi_{\tau,\tau+1}$, are the period-to-period pricing kernels.

$$S_t = E_t \{ \phi_{t,t+1} E_{t+1} [\phi_{t+1,t+2} E_{t+2} (\phi_{t+2,t+3} B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3} x_{t+3})] \},$$

The futures price $H_{t,t+3}$ is the time- t value of $\frac{x_{t+3}}{B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3}}$

$$S_t = E_t \{ \phi_{t,t+1} E_{t+1} [\phi_{t+1,t+2} E_{t+2} (\phi_{t+2,t+3} B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3} \frac{x_{t+3}}{B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3}})] \}$$

$$H_{t,t+3} = E_t \{ \phi_{t,t+1} E_{t+1} [\phi_{t+1,t+2} E_{t+2} (\phi_{t+2,t+3} x_{t+3})] \}$$

A Great Result

$$H_{t,t+3} = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\}$$

or,

$$\begin{aligned} H_{t,t+3} &= E_t^Q\{E_{t+1}^Q[E_{t+2}^Q(x_{t+3})]\} \\ &= E_t^Q(x_{t+3}) \end{aligned}$$

Futures Price: Interpretation

- Under risk neutrality, futures = expected payoff
- Futures price has 'martingale property' under Q (risk-neutral) measure
- Compounding effect offsets stochastic discounting
- Under risk neutrality, forward \neq expected payoff

The Forward Price

the forward price of \mathbf{x}_{t+3} is the spot price of an asset paying $\mathbf{x}_{t+3}/B_{t,t+3}$ at time $t + 3$.

$$F_{t,t+3} = E_t^Q \left(\frac{B_{t,t+1} B_{t+1,t+2} B_{t+2,t+3} \mathbf{x}_{t+3}}{B_{t,t+3}} \right).$$

$$F_{t,t+T} = E_t^Q \left[\prod_{\tau=1}^T \frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}} \mathbf{x}_{t+T} \right].$$

$$b_{t,t+T} = \prod_{\tau=1}^{T-1} \frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}}.$$

$$E_t^Q(b_{t,t+T}) = 1$$

The Forward Price

$$F_{t,t+T} = E_t^Q(b_{t,t+T}x_{t+T}).$$

$$F_{t,t+T} = E_t^Q(b_{t,t+T})E_t^Q(x_{t+T}) + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

$$F_{t,t+T} = H_{t,t+T} + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

The Forward Price: Interpretation

- Forward price = Futures price + covariance term
- Covariance may be positive for many assets
- Covariance is positive for bonds
- For bonds, Forward price $>$ Futures price

An Example: Lognormal Variables

Assume:

- \mathbf{x}_{t+T} is lognormal
- $\phi_{t,t+T}^* = \phi_{t,t+1}\phi_{t+1,t+2}\phi_{t+2,t+3}\dots$ is lognormal
- $\mathbf{b}_{t,t+T}$ is lognormal

If \mathbf{x}, \mathbf{y} are lognormal with

$$\mu_x, \sigma_x$$

$$\mu_y, \sigma_y$$

$$\sigma_{x,y}$$

Then

$$E(xy) = E(x)E(y)e^{\sigma_{x,y}}$$

Futures price:

$$\begin{aligned}H_{t,t+T} &= E(x\phi) \\&= E(x)E(\phi)e^{\sigma_{x,\phi}} \\&= E(x)e^{\sigma_{x,\phi}} \\&= e^{\mu_x + \frac{1}{2}\sigma_x^2}e^{\sigma_{x,\phi}} \\&= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,\phi}}\end{aligned}$$

Forward price

$$\begin{aligned}F_{t,t+T} &= E(xb\phi) \\&= E(x)E(b\phi)e^{\text{cov}(\ln x, \ln b\phi)} \\&= E(x)e^{\text{cov}(\ln x, \ln b) + \text{cov}(\ln x, \ln \phi)} \\&= e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x,b} + \sigma_{x,\phi}} \\&= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,b} + \sigma_{x,\phi}}\end{aligned}$$

The Forward and Futures Prices: Lognormal Variables

Then:

- Futures price:

$$H_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_x \phi}.$$

- Forward price

$$F_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_x \phi + \sigma_x b}.$$

The Forward-Futures Bias: Lognormal Variables

-

$$F_{t,t+T} = H_{t,t+T} e^{\sigma_{xb}}$$

-

$$\sigma_{xb} = \sigma_x \sigma_b \rho_{xb}$$

- For Bonds

$$\rho_{xb} > 0$$

- σ is non-annualised
- Forward-futures bias increases with maturity of futures

The Forward-Futures Bias: Further Results

- Interest rates: inversely related to bond prices
- Section 6.5: define futures rate by:

$$H_{t,t+T} = e^{-h_{t,t+T}}.$$

- Define forward rate by:

$$F_{t,t+T} = e^{-f_{t,t+T}}.$$

-

$$\ln \left(\frac{F_{t,t+T}}{H_{t,t+T}} \right) = h_{t,t+T} - f_{t,t+T} = \sigma_{xb},$$

- Futures rate $>$ Forward rate
- Important for interest-rate modelling (see ch 7)

The Forward-Futures Bias: Further Results

- For contingent claims (options) bias is magnified
- On LIFFE, Sydney, options traded on futures basis
- See section 6.6

The Forward-Futures Bias: Conclusions

- Forward-Futures bias depends on covariance of asset price with the bond roll-up factor
- Bias is more significant for bond futures
- Forward price $>$ Futures price
- Interest rates: Forward rate $<$ Futures rate
- Bias increases with maturity of futures contract
- Bias is magnified in the case of futures on options