

Background Risk and Trading in a Full-Information Rational Expectations Economy

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Abstract

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In this paper, we assume that investors have the same information, but trade due to the evolution of their non-market wealth. In our formulation, investors rebalance their portfolios in response to changes in their expected non-market wealth, and hence trade. We assume an incomplete market in which risky non-market wealth is non-hedgeable and independent of market risk, and thus represents an additive background risk. Investors who experience positive shocks to their expected wealth buy more stocks from those who experience less positive shocks.

1 Introduction

It has long been a challenge for financial economists to explain trading in the context of rational expectations asset pricing models. For example, in the complete markets Arrow-Debreu model, agents choose state-contingent claims on the initial date, but do not trade at subsequent dates, since they have already purchased claims that hedge against various future outcomes; thus, there is no need for them to adjust their portfolio holdings as the state of the world is revealed. This inability to explain trading in a rational model flies in the face of evidence that there is a large volume of trading in various securities: bonds, stocks, and increasingly in various types of contingent claims, such as options and futures contracts.

Several attempts have been made in the literature in the past to explain trading by relaxing some of the assumptions of completeness of markets and information available to agents in the economy. One possibility is that when investors have asymmetric information, this gives them an incentive to trade in order to profit from that information. However, as Grossman and Stiglitz (1980) point out, the mere act of trading reveals the information possessed by a particular agent and this gets reflected in market prices. While there may be some “sand in the gears” introduced if the process of expectations formation is noisy, the central intuition that prices reflect private information still prevails, reducing the motivation to trade substantially.

This argument was taken one step further by Milgrom and Stokey (1982) who argue that when the agents begin with a Pareto optimal allocation relative to their prior beliefs, they do not trade upon receiving private information, even at equilibria that are less than fully revealing, since “the information conveyed by price changes swamps each traders private information.” This surprisingly general result arises because if the initial allocation is Pareto optimal, there is no valid insurance motive for trading. The willingness of other traders to take the opposite side implies at least to one trader that his own bet is unfavorable. Hence no trade is acceptable to all traders. The Milgrom and Stokey propositions rely on two crucial assumptions: a) that it is common knowledge that when a trade occurs it is feasible and acceptable to all agents, and b) the agents beliefs are concordant, i.e., that they agree about how the information should be interpreted.

Another strand of the literature that has provided a motivation for trading is on market micro-structure, most prominently by Kyle (1985) and Glosten and Milgrom (1985). These models try to explain the bid-offer spread in markets by appealing to asymmetric information. However, a crucial assumption in such models is the existence of noise traders, who trade for liquidity reasons, and these are not explicitly modeled. Furthermore, it is unclear why in such models, investors trade for liquidity reasons in risky securities such as stocks,

rather than trading bonds, unless some market imperfection is assumed. In the Milgrom and Stokey sense, it must be the case that the allocation in these models is not ex-ante Pareto optimal, and/or that the beliefs are not concordant.

The broad conclusion from the information-based literature on trading is that the Milgrom and Stokey “no-trade” result will obtain, unless there is some market imperfection, significant deviation from rational expectations equilibria or an exogenous reason to trade, such as liquidity motivations.

In this paper, we explore an alternative motivation for trading, which is the existence of non-marketable wealth. Non-marketable wealth may take many forms, but the most obvious example is wealth arising from labor income. Human capital, which is the value of future labor income, has been shown in many studies, both theoretical and empirical, to have an influence on portfolio demand. Another example is housing wealth, which is a significant component of the portfolios of households. Again, there is an extensive literature documenting how housing wealth affects portfolio choice and, in turn, feeds back on to the equilibrium prices of traded assets. The effect of non-market wealth is that it alters the agents’ demand for the traded assets. An early example of this distortion is the work of Bodie, Merton and Samuelson (1992) in the context of non-stochastic, positive non-marketable wealth for an agent with constant relative risk aversion. They show that this agent acts much like another agent with a lower, but increasing relative risk aversion.

The problem gets more complex when the non-marketable wealth has stochastic properties. There is an extensive literature on background risk that studies the portfolio behavior of agents with such non-marketable wealth, whose future cash flows are also stochastic. For most common utility functions, the existence of background risk makes agents more risk averse and hence reduces their demand for risky securities.¹ The natural question is how the changes in the agents’ portfolio decisions affect the portfolio demand and sharing rules of the marketable securities in equilibrium, a problem first analyzed by Franke, Stapleton and Subrahmanyam (1998) [FSS].

We extend this framework to consider a multi-period version of the FSS framework. Following the outcome of the background risk in the intermediate period, agents adjust their holdings of the marketable securities, to be in line with their new level of derived risk aversion in the presence of the updated distribution of background wealth. If the outcomes of the background risk are heterogeneous across agents, it creates a motivation for trading, as different agents may wish to adjust their portfolio holdings in opposite directions. We explore this simple intuition formally for investors with constant relative risk aversion in our analysis.

¹See, for example, Gollier and Pratt (1996), Kimball (1993) and Eekhoudt, Gollier and Schlesinger (1996).

Section 2 presents the set up of the model and derives the portfolio demand for traded state-contingent claims. Section 3 describes the evolution of the background risk over time. Section 4 derives optimal demand in the special case where all uncertainty of background risk is resolved at time 1. Section 5 generalizes the results using an approximation. Section 6 presents our conclusions.

2 A Single-Period Model

In this section we derive the optimal demand for contingent claims for agents in a single-period equilibrium economy. The results will provide the basic building block for our multi-period trading model in later sections. The set-up of the model is similar to that in Franke, Stapleton and Subrahmanyam (1998) [FSS].² As in FSS, we assume that all agents maximize the expected utility of wealth, w at the end of a single period. For agent i , $w_i = x_i + y_i$, where x_i is a set of claims on a single aggregate market cash flow X_a and y_i is the nonmarketable income, e.g. labor income. In general, the non-marketable income $y_i = a_i + e_i$, where a_i is a constant representing the expected value of non-market income, and e_i is an independent, zero-mean background risk. Each agent solves the following maximization problem:

$$\max_{x_i} E_{X_a}[E_e[u_i(w_i)]], \text{ s.t. } E[\phi(X_a)x_i] = E[\phi(X_a)\hat{x}_{i0}], \quad (1)$$

given an initial endowment of x , \hat{x}_{i0} . In (1), $\phi(X_a)$ is the forward pricing kernel. The budget constraint states that the forward price of the chosen portfolio of claims has to equal the forward value of the endowed claims. In FSS, agents have utility functions $u_i(w_i)$ which belong to the HARA class, excluding the exponential function. Here, we assume essentially the same setup with

$$u_i(w_i) = \frac{w_i^{1-\gamma_i}}{1-\gamma_i}. \quad (2)$$

where γ_i is the coefficient of relative risk aversion. Utility for wealth is a power function, exhibiting constant relative risk aversion, but the derived utility for x_i is of the HARA form, when the background risk e_i does not exist.³

²However, we cannot simply use the results in FSS, since in that paper they do not solve for the Lagrangian multipliers, see λ_i below. Hence, their results show that some investors buy, and some sell contingent claims but do not show how many claims are bought or sold.

³Utility is of the Hypobolic Absolute Risk Averse (HARA) class if

$$u_i(w_i) = \frac{(w_i + a_i)^{1-\gamma_i}}{1-\gamma_i}$$

Let λ_i be the Lagrangian multiplier associated with the budget constraint of investor i . The Lagrangian multiplier is then:

$$L = u_i(w_i) + \lambda_i(E[\phi(X_a)\hat{x}_{i0}] - E[\phi(X_a)x_i]). \quad (3)$$

It follows that the first order condition of the optimization problem is:

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] = \lambda_i\phi(X_a). \quad (4)$$

Following Kimball (1990), we can define the precautionary premium $\psi_i(x_i)$ by the relation

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] \equiv [x_i + a_i - \psi_i(x_i)]^{-\gamma_i} \quad (5)$$

Hence $(x_i + a_i - \psi_i)^{-\gamma_i}$ is the certainty equivalent of $E_e(x_i + a_i + e_i)^{-\gamma_i}$. Note that ψ_i itself will be a function of x_i and also depends on the distribution of e_i . More specifically, the function $\psi(\cdot)$ is decreasing and convex. The above result differs slightly from FSS in that we allow the mean of the background risk to be non-zero. This difference is essential for our setting because in the dynamic case, analyzed in sections 3 and 4, the mean of the background risk will be non-zero after the initial date.

Substituting the above certainty equivalence into the first order condition:

$$[x_i + a_i - \psi_i]^{-\gamma} = \lambda_i\phi(X_a), \quad (6)$$

and it follows that the demand for contingent claims is given by:

$$x_i = (\lambda_i)^{-1/\gamma_i}\phi(X_a)^{-1/\gamma_i} - a_i + \psi_i. \quad (7)$$

The optimal demand consists of three separate parts. The first term is the demand if the expected non-marketable income is zero and the precautionary premium is also zero (i.e. the background risk is zero). When the expected non-marketable income is positive (negative) the demand is reduced (increased) in each state to compensate. This explains the second term. The third term adjusts for the effect of the background risk.

To obtain the optimal demand, we need to solve for λ_i and the pricing kernel $\phi(X_a)$. It turns out that it is more convenient to use the per capita term X , instead of the aggregate X_a . Using the market clearing condition $\frac{1}{I}\sum_i x_i = X$, where I is the number of agents and assuming $\gamma_i = \gamma$ for all i , we have:⁴

$$X = \lambda^{-1/\gamma}\phi(X)^{-1/\gamma} - A + \psi, \quad (8)$$

⁴One could still keep the general form of different γ_i at this stage, but the resulting expression will be quite complicated.

where

$$\psi = \frac{1}{I} \sum_i \psi_i, \quad (9)$$

$$A = \frac{1}{I} \sum_i a_i, \quad (10)$$

$$\lambda^{-1/\gamma} = \frac{1}{I} \sum_i \lambda_i^{-1/\gamma}. \quad (11)$$

Note that the aggregate ψ is a function of the state indexed by X and depends also on the distribution $\{e_i\}_{i=1,\dots,n}$. This is essentially a representative agent version of equation (7), assuming that all the γ_i 's are the same. Note also that we do *not* assume that the background risks are identical across all agents. Indeed, in the subsequent analysis we will use the fact that a_i and ψ_i vary across agents to create an incentive to trade. Initially, the agents are all identical in terms of their *original* risk aversion. However, the realization of the background risks can differ and consequently the *derived* risk aversion can be different. This is the basic intuition behind the trading in our model.

Solving (8) for ϕ we find

$$\phi(X) = (X + A - \psi)^{-\gamma} \lambda^{-1}. \quad (12)$$

Now, substituting the solution of x_i in (7) above back into the individual budget constraint

$$E[\phi(X)x_i] = E[\phi(X)\hat{x}_{i0}],$$

it follows that:

$$\begin{aligned} E[\phi(X)\hat{x}_{i0}] &= E\{\phi(X)[\lambda_i^{-1/\gamma}\phi(X)^{-1/\gamma} - a_i + \psi_i]\} \\ &= \lambda_i^{-1/\gamma} E[\phi(X)^{1-\frac{1}{\gamma}}] - E[\phi(X)a_i] + E[\phi(X)\psi_i]. \end{aligned}$$

Then, we obtain the following:

$$\lambda_i^{-1/\gamma} = \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \quad (13)$$

$$\text{or } \lambda_i = \left\{ \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \right\}^{-\gamma} \quad (14)$$

Hence, the optimal individual investor demand is (using equation (12)):

$$x_i = \frac{E[\phi(X)(\hat{x}_{i0} + a_i - \psi_i)]}{E[\phi(X)^{1-\frac{1}{\gamma}}]} \phi(X)^{-\frac{1}{\gamma}} - a_i + \psi_i(x_i) \quad (15)$$

$$= \frac{E[(X + A - \psi)^{-\gamma}(\hat{x}_{i0} + a_i - \psi_i)]}{E[(X + A - \psi)^{1-\gamma}} (X + A - \psi) - a_i + \psi_i. \quad (16)$$

The expression for the demand for contingent claims in (16) is complex. If there were no background risk for all investors, ψ would be zero and x_i would be linear in X . However, in general, both ψ and ψ_i are convex functions implying a non-linear demand function. Also, the optimal demand is implicit, since ψ_i is a function of x_i for each i .

Again, the optimal individual demand consists of three parts. The first term is linear in per capita market cash flow. The coefficient depends on the expectation of the individual precautionary premium. The second term is the adjustment for the non-zero expected background risk, a_i . The third term is the adjustment for individual precautionary premium.

3 The Evolution of Background Risk Over Time

So far, we have assumed that agents face a background risk e_i which is resolved at the end of a single period. As in FSS, e_i has a zero mean and is independent of the market cash flow, X . We now introduce a multiperiod model in which the risk, e_i , evolves over time. This is required to study trading volume in the following sections, since trading is essentially an intertemporal issue.

There are three dates, $t = 0, 1, 2$ in the model. These are represented in Figure 1 below.

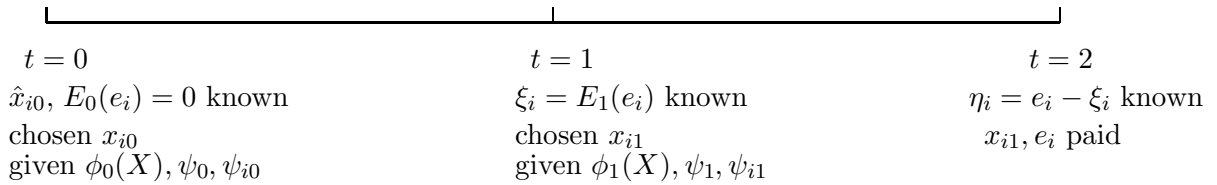


Fig 1. The timeline

At time $t = 0$, each agent is endowed with \hat{x}_{i0} , which is a portfolio of marketable contingent claims. Also, at $t = 0$ each agent knows about the distribution of the background risk e_i , which will be fully revealed at $t = 2$. The agent chooses a portfolio of marketable contingent claims, at time 0, x_{i0} , to maximize the expected utility of the wealth at time $t = 2$. The maximization is given the pricing kernel $\phi_0(X)$ and the precautionary premium ψ_{i0} . Note that, all the payoffs, which include the payoff from the marketable contingent claims and the background risk e_i , are at $t = 2$.

At time $t = 1$, the agent receives information about her background risk, ξ_i , and revises her expectation of e_i to $E_1(e_i) = \xi_i$. Given this information and the revised distribution of the background risk in light of the information, e_i , she chooses a new portfolio of contingent claims, given an updated pricing kernel $\phi_1(X)$ and a revised precautionary premium, ψ_{i1} . Then, at time $t = 2$, the agent receives more information about her background risk, η_i , and both payments x_{i1} and e_i are paid to the agent.

Note that, in this model, the agent knows about part of the final payoff from the background risk at $t = 1$. Thus, ξ_i is the conditional expectation at $t = 1$ of the background risk at $t = 2$. However we should emphasize that even though the agent knows about ξ_i , she cannot use it directly to trade the contingent claims because the risk is non-marketable. However, the agent does change her optimal portfolio holdings of marketable claims at $t = 1$, given the new information. This is the trading generated in this model.

We next assume that there are two groups of agents, which are indexed as $i = m, n$. Without loss of generality, we assume that the two groups are of equal size. For simplicity, we also assume that *ex ante* at time $t = 0$, the distributions of e_i are the same for the two groups $i = m, n$.

The trading that takes place at $t = 1$ depends on the cross-sectional realization of ξ_i across the agents. If it happens that the outcome ξ_i is the same for both groups of investors, there will be no incentive for the two groups to trade with each other. However, if the realizations of ξ_i 's are different for the two groups, then there will be trade between them. We proceed by first considering a special case where the precautionary premia at $t = 1$ are zero for all investors. This is the case where there is full resolution of uncertainty about e_i at $t = 1$.

4 A Special Case: Full Resolution of Background Risk at Time 1

In this section, we investigate the case where all the uncertainty of ε_i is resolved at $t = 1$. As discussed above, in the general case, the demand for contingent claims is an implicit function.

This is due to the fact that the demand is a function of the precautionary premium, but the precautionary premium itself is a function of the demand. However, in the special case where all the uncertainty of the background risk ε_i is resolved at $t = 1$, the precautionary premium, $\psi_i(x_i)$, is zero at time 1. So, in this case, there is an explicit solution for the optimal demand at time 1.

At time 0, all the investors are identical, and only differ in the resolution of the uncertainty of e_i . Since the investors are identical at $t = 0$ and since ε_i has the same distribution for all i , the investors must hold the same portfolios at $t = 0$. That implies that the initial demand $x_i = X$, since X is the average allocation of claims across investors.

Now, let the realization of ξ_i take on the values: $\xi_i = a_k, k = 1, \dots, K$. For the two groups, we assume $(\xi_m, \xi_n) = (a_m, a_n)$, where $a_m, a_n \in \{a_1, a_2, \dots, a_K\}$. Using (16), with $\psi_i = 0$ and $\hat{x}_{i0} = X$ we have the demand for agent m :

$$x_{m1}^* = \frac{E_1[(X + A)^{-\gamma}(X + a_m)]}{E_1[(X + A)^{1-\gamma]}}(X + A) - a_m \quad (17)$$

which can be written

$$x_{m1}^* = X + (a_m - A)\beta(X + A) - (a_m - A), \quad (18)$$

where

$$A = \frac{ma_m + na_n}{m + n} \quad (19)$$

is the aggregate, average shock and

$$\beta = \frac{E_1[(X + A)^{-\gamma]}}{E_1[(X + A)^{1-\gamma]}} \quad (20)$$

is a constant.⁵ The demand of m -type agents is made up of three terms. The first term is the agent's initial allocation, X . Note that if the shock a_m equals the aggregate average shock, A , then m -type agent's demand is simply the initial allocation and she does not trade. The second and third terms are dependent on the divergence in the background risk between each m -type agent m and the market average. The second term is a linear demand for claims on X . It depends on $a_m - A$, the difference between the shock to m -type agent's shock, a_m , and the aggregate average shock, A . Also, the coefficient β depends on A , and is declining as A increases. The third, constant term, $a_m - A$, represents a demand for risk-free income which balances the increase/decrease in expected non-market wealth.

⁵For an agent with background income equal to the average A , β is the expected marginal (derived) utility scaled by the expected (derived) utility.

Similarly, the demand for agent n is

$$x_{n1}^* = X + (a_n - A)\beta(X + A) - (a_n - A), \quad (21)$$

We now define the trading in contingent claims for an m -type agent, m , z_{m1} , as the absolute value of the difference between the demand $x_{m,1}$ and her initial allocation X . From (18)

$$z_{m1} \equiv |x_{m1}^* - X| = |(a_m - A)\beta(X + A) - (a_m - A)|. \quad (22)$$

Similarly, the trading for an n -type agent is

$$z_{n1} \equiv |x_{n1}^* - X| = |(a_n - A)\beta(X + A) - (a_n - A)|. \quad (23)$$

The optimal demand and trading of the two types of agents are functions of the exogenous variables (a_m, a_n, m, n) . However, an alternative and more economically meaningful set of exogenous variables are the average expected background income A , the difference between the expected background incomes of the two groups of agents, and the relative number of m agents, m/n .

For convenience, we therefore define:

$$\rho \equiv \frac{m}{n}. \quad (24)$$

and

$$\Delta \equiv a_m - A \quad (25)$$

Given these definitions we have

Proposition 1 (Demand for Contingent Claims: Full Resolution Case) *Assuming full resolution of the uncertainty of background risk at time 1,*

(a) *the demand for contingent claims for agents in groups m and n is given by:*

$$x_{m1}^* = X + \Delta\beta(X + A) - \Delta \quad (26)$$

$$x_{n1}^* = X - \rho\Delta\beta(X + A) + \rho\Delta, \quad (27)$$

(b) *and the trading of the agents is given by:*

$$z_{m1} = |\Delta\beta(X + A) - \Delta| = |\Delta| \cdot |\beta(X + A) - 1| \quad (28)$$

$$z_{n1} = |\rho\Delta\beta(X + A) - \rho\Delta| = \rho|\Delta| \cdot |\beta(X + A) - 1|. \quad (29)$$

Proposition 1 follows directly from equations (18) and (21), after substituting (24) and (25). Proposition 1, part *a.* states the $t = 1$ demand for state contingent claims on X for agents in groups m and n . $x_{m,1}^*$ and $x_{n,1}^*$ are the demands in the case of full resolution of uncertainty. The demand for claims in different states depends on Δ , which reflects the difference between the background income realizations of the two groups of agents $a_m - A$, and A , the average background income realization, and ρ , the relative number of agents in each group.

Proposition 1, part *b.* shows the implications of the optimal demands for the trading of contingent claims. The amount of trading of claims depends upon the heterogeneity of the two groups of agents, measured by $|\Delta|$, and ρ .

We now analyze the comparative static properties of the equilibrium in the following three corollaries. First, we have:

Corollary 1 [Changes in Heterogeneity and Trading]

Holding the average background income, A , and the relative numbers of agents, ρ fixed:

(a) *the effect of a change in the difference, Δ , is given by:*

$$\frac{\partial x_{m1}^*}{\partial \Delta} = \beta(X + A) - 1 \quad (30)$$

$$\frac{\partial x_{n1}^*}{\partial \Delta} = -\rho(\beta(X + A) - 1), \quad (31)$$

(b) the effect of a change in the absolute value of the heterogeneity, $|\Delta|$, on the trading is given by

$$\frac{\partial z_{m1}}{\partial |\Delta|} = |\beta(X + A) - 1| \geq 0 \quad (32)$$

$$\frac{\partial z_{n1}}{\partial |\Delta|} = \rho |\beta(X + A) - 1| \geq 0, \quad (33)$$

The corollary follows directly from the optimal demands and trading in Proposition 1. Part a shows the effect of a change in the difference $\Delta = a_m - A$. When the heterogeneity across agents increases (while holding the average A fixed), the m -type agents will demand more claims on the high states of X , i.e. $\partial x_{m1}^*/\partial \Delta > 0$ for high X , and demand less on the low states of X . Hence, m -type agents buy more claims on the high states and sell more claims on the low states, with n -type agents doing the opposite.

The intuition behind this result is as follows. First, suppose that $\Delta < 0$ and decreases. The m -type agents are the poorer type and their background risk, $a_m < 0$, decreases. The reaction of m -type agents is to buy more claims on low states and sell more claims on high states. Conversely, if Δ increases, the m -type agents will sell more claims on low states and buy more claims on high states. Alternatively, suppose that $\Delta > 0$ and increases. In this case, the poorer n -type agents buy more claims on low states and sells more claims on high states. Hence, the richer m -type agents again sell more claims on low states and buys more claims on high states as Δ increases.

Corollary 1 part a) also implies that there exists a critical contingent claim X_1^* in which no trade takes place. This “breakeven state” is given by

$$X_1^* \equiv \frac{1}{\beta} - A. \quad (34)$$

If there is an increase in Δ , for those states $X > X_1^*$, m -type agents will demand more claims; and for those $X < X_1^*$, m -type agents will demand less claims.

The second part of the Corollary, 1 b), shows that trading of the agents increases when the heterogeneity increases. Heterogeneity in the realization of expected background income is the key motivation for agents to trade in this model. Note that in this case, the aggregate economy is assumed to be the same, since we hold A , the average expected background income, fixed.

Proposition 1 and Corollary 1 are illustrated in Figure 1, panel A.⁶ In this case, for simplicity, we choose $A = 0$ and $\rho = 1$. Also $\Delta = a_m - A < 0$. The solid line shows the excess demand

⁶The examples in Figures 1 and 2 are based on a simple 12-state case. Details are provided in Appendix.

(compared to the endowment) of an m -type agents when $\Delta = -1$. The agent buys claims in the low states and sells claims in the high states. The dotted line represents the (symmetric) excess demand of n -type agents. If the difference falls to $\Delta = -3$, the demand of m -type agents changes to that shown by the dashed line. The m -type agents increase their demand for claims in the low states and increase sales of claims on the high states. This increased hedging behavior is matched by the n -type agents who reduce their demand for claims on income in the low states and increase demand for claims in the high states. The dotted-dashed line shows the excess demand of the n -type agents.

We now assume that there exists some heterogeneity across the two types of agent and examine the effect of a change in the average background income realization, A on the trading pattern of the agents. We have the following:

Corollary 2 [Changes in Average Background Income and Trading]

Holding the difference, Δ , and the relative numbers of agents, ρ fixed:

- (a) *the effect of a change in the average background income, A , on the demand for contingent claims is given by:*

$$\frac{\partial x_{m1}^*}{\partial A} = \Delta\beta[1 - (\gamma\delta + (1 - \gamma)\beta)(X + A)], \quad (35)$$

$$\frac{\partial x_{n1}^*}{\partial A} = -\rho\Delta\beta[1 - (\gamma\delta + (1 - \gamma)\beta)(X + A)], \quad (36)$$

where⁷

$$\delta \equiv \frac{E(X + A)^{-1-\gamma}}{E(X + A)^{-\gamma}} \quad (37)$$

- (b) *holding Δ, ρ fixed, if A changes, then the effect on the trading of the agents is:*

$$\frac{\partial z_{m1}^*}{\partial A} = \text{sign}(\Delta(X - X_1^*))\Delta\beta[1 - (\gamma\delta + (1 - \gamma)\beta)(X + A)], \quad (38)$$

$$\frac{\partial z_{n1}^*}{\partial A} = -\text{sign}(\Delta(X - X_1^*))\rho\Delta\beta[1 - (\gamma\delta + (1 - \gamma)\beta)(X + A)]. \quad (39)$$

Corollary 2a) shows the effect of a marginal change in average background income on the demand for contingent claims of the two types of agent. To interpret the derivative in (35) note that δ in (37) is positive, since $X + A$ is positive. Moreover, comparing δ in (37) with

⁷The term δ is analogous to the β definition earlier in equation (20), except that δ uses the marginal (derived) utility function instead of the (derived) utility.

β in (20) we have $\delta \geq \beta$.⁸ It follows that $\gamma\delta + (1-\gamma)\beta > 0$. Hence, for $\Delta < 0$, the derivative is negative for small X and positive for large X .

The intuition for this result is as follows. Suppose that $\Delta < 0$ and A falls. Since the m -type agents are poorer in prospects, they are relatively more affected by the fall in average expected background income and buy claims on low states from the n -type agents who are relatively less affected by the shock. They buy these low-state claims and sell claims on the higher states.

Another way of interpreting the result is in terms of a wealth effect and a substitution effect of the change in A . With an increase in A , the wealth effect for all agents is the same, the greater the wealth, the lower the risk aversion, pushing up the demand for all claims. However, the substitution effect is different for the two groups, depending on which group experiences the bigger *relative* increase in wealth. With $\Delta < 0$ and hence $a_m < a_n$, the effect on a_m is stronger, resulting in an increase in demand for low state claims from m -type agents. This is matched by a decrease in the demand for such claims from n -type agents.

[The description and the panels in the figures have to be rearranged so that they are in the same sequence as in the corollaries.]

Corollary 2b) summarizes the effect of a small change in average background income on the trading activity of the two types of agent. For $X < X_1^*$, the term $\text{sign}(\Delta(X - X_1^*))$ is positive (since $\Delta < 0$). The final term in square brackets is positive for small X and negative for larger X . Hence, the trading in this region may increase or decline. However, for $X > X_1^*$, the sign of $(\Delta(X - X_1^*))$ is negative and it follows that trading of claims in this region increases.

Corollary 2a) is illustrated in Figure 1, panel C, for the case where the average expected background risk of agents changes from $A = 0$ to $A = -1$. When $A = 0$, the demands for claims of the m -type and n -type agents are shown by the solid line and the dotted line respectively. When A changes to $A = -1$, the demands are shown by the steeper dashed and dotted-dashed lines, respectively. With $A = 0$, the cross-over state at which no trade takes place is $X_1^*(0)$. With $A = -1$, the cross-over state at which no trade takes place falls to $X_1^*(-1)$. The effect of the fall in A is to steepen the linear demand for contingent claims. The m -type agents require more claims on the low states and these are supplied by the relatively rich n -type agents.

The effect on the level of trading of such a finite change in A is more complex, however, as illustrated in panel D of Figure 1. There are two levels of X at which the trading level is unchanged as A changes. These are denoted X_2^* and X_2^{**} . From the Figure, we see that

⁸This follows from the Cauchy-Schwartz inequality, as shown in the Appendix.

for $X < X_2^*$ and $X > X_2^{**}$, trading increases with a fall in A . However, for $X_2^* < X < X_2^{**}$ trading falls, with a reduction in A .

Finally, we study the comparative statics results from the change in the relative number of agents: $\rho = m/n$. We have the following:

Corollary 3 [Changes in the Relative Numbers of Agents and Trading]

Holding the average expected background income, A , and the difference, Δ fixed, i.e. holding A, a_m fixed:

(a) *The effect of a change in the relative number of agents, ρ , is given by:*

$$\frac{\partial x_{m1}^*}{\partial \rho} = 0 \quad (40)$$

$$\frac{\partial x_{n1}^*}{\partial \rho} = \Delta(1 - \beta(X + A)). \quad (41)$$

(b) *The effect of a change in ρ on the trading of m -type agents is zero, while the trading of n -type agents will be affected as follows:*

$$\frac{\partial z_{n1}^*}{\partial \rho} = |\Delta(1 - \beta(X + A))| > 0. \quad (42)$$

The intuition for this trading behavior is as follows. When a_m, A are fixed, agent m 's demand will not change. However, since the proportion $\rho = \frac{m}{n}$ increases, there must be fewer n -type agents per m -type agent. Hence, the trading of each n -type agent increases.

The result in Corollary 3a) is illustrated in Figure 1, panel B. The solid line and the dotted line shows the excess demand for agents of type m and n respectively, when $\rho = 1$. When ρ increases, the steepness of the n -type agent's demand curve increases to that shown by the dotted-dashed line for the case where $\rho = 2$.

5 The General Case: Partial Resolution of Uncertainty at $t = 1$

As we saw earlier, in the general case where there is unresolved background risk at time 1, the optimal demand, x_i , cannot be solved analytically in closed-form. In the previous

section, the problem above was solved by considering a special case where all background risk was resolved at time 1, and the precautionary premium, ψ_i , was zero. We now analyze the general case, using an alternative approach involving an approximation for the precautionary premium.

5.1 An Approximation for the Precautionary Premium.

At time 1, the residual background risk is η_i with variance σ_η^2 , which we will assume is the same across agents.⁹ From Gollier (2001), the precautionary premium, ψ_i , can be approximated by¹⁰

$$\psi_i(x_i) \approx \frac{1}{2} \left(-\frac{u_i'''(x_i + a_i)}{u_i''(x_i + a_i)} \right) \sigma_\eta^2 \quad (43)$$

for a small background risk. This is analogous to the Arrow-Pratt approximation for the risk premium, using the marginal utility function instead of the original utility function. For the special case where $u(\cdot)$ is CRRA analyzed earlier, we have

$$\psi_i(x_i) \approx \frac{(1 + \gamma)\sigma_\eta^2}{2(x_i + a_i)}. \quad (44)$$

Thus, we have an approximate solution for ψ_i as a function of x_i .¹¹

⁹Generalizing this to the case of heterogeneity of variances across agents adds notational complexity without yielding any additional insights.

¹⁰See Gollier (2001), p. 237.

¹¹As we can see, the approximation satisfies all the properties for the precautionary premium, ψ_i , as stated in FSS:

$$\begin{aligned} \psi_i &> 0, & \frac{\partial \psi_i}{\partial x} &< 0, & \frac{\partial^2 \psi_i}{\partial x^2} &> 0, \\ \frac{\partial \psi_i}{\partial \sigma} &> 0, & \frac{\partial^2 \psi_i}{\partial \sigma \partial x} &< 0, & \frac{\partial^3 \psi_i}{\partial \sigma \partial x^2} &> 0. \end{aligned}$$

Also, the approximation has additional implications with respect to the change in the expectation of the background risk a_i :

$$\frac{\partial \psi_i}{\partial a_i} < 0, \quad \frac{\partial^2 \psi_i}{\partial a_i^2} > 0$$

Finally, the cross derivatives with respect to σ and a_i are similar to those for σ and x_i .

5.2 Optimal demand given the approximation for ψ_i

From equation (16), the optimal demand for members of the two groups of agents is:

$$x_{m1} = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_m - \psi_{m1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) - a_m + \psi_{m1} \quad (45)$$

$$x_{n1} = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_n - \psi_{n1})]}{E_1[(X + A - \psi_1)^{1-\gamma}]}(X + A - \psi_1) - a_n + \psi_{n1}, \quad (46)$$

where

$$\begin{aligned} \psi_{m1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2(x_{m1} + a_m)} \\ \psi_{n1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2(x_{n1} + a_n)} \\ \psi_1 &= \frac{1}{1 + \rho}(\rho\psi_{m1} + \psi_{n1}) \end{aligned}$$

The optimal demands of the two types of agent are implicit in equations (45) and (46). However, in the appendix we show, using approximations, that the following proposition holds in the general case:

Proposition 2 (Demand for Contingent Claims: General Case) *The optimal demand of the two groups of agents in general is:*

$$x_{m1} = x_{m1}^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1m}(X + A) + B_{2m}\frac{1}{(X+A)} \right) \quad (47)$$

$$x_{n1} = x_{n1}^* + \frac{(1+\gamma)\sigma_\eta^2}{2} \left(B_{1n}(X + A) + B_{2n}\frac{1}{(X+A)} \right), \quad (48)$$

where

$$\begin{aligned} B_{1m} &= \frac{\beta\delta\Delta}{(1 + \Delta\beta)(1 - \rho\Delta\beta)} \{ \beta(2 + (1 - \rho)\Delta\beta) + \gamma(1 + (1 - \rho)\Delta\beta)(\theta - \beta) \}, \\ B_{2m} &= \frac{-\Delta\beta[2 + (1 - \rho)\Delta\beta]}{(1 + \Delta\beta)(1 - \rho\Delta\beta)}, \end{aligned}$$

and¹²

$$\theta = \frac{E_1[(X + A)^{-2-\gamma}]}{E_1[(X + A)^{-1-\gamma}]}$$

and when $\rho = 1$, $B_{1n} = -B_{1m}$ and $B_{2n} = -B_{2m}$.

¹²Again θ is analogous to δ and β , in that we start with the second derivative of the (derived) utility function instead of the utility function as the basis.

Proof See Appendix.

The main properties of the optimal demand and trading can be analyzed using x_{m1}^*, x_{n1}^* , the full-resolution demands for agent m and n respectively, as the base case. Recall that in the special case of full resolution, x_{m1}^* has the property that when $\Delta > 0$, $x_{m1}^* - X$, the amount of trading, is an upward sloping linear function of states X ; and when $\Delta < 0$, $x_{m1}^* - X$ is a downward sloping linear function of states X .

The additional effect in the general case comes from two terms: one from the linear term $B_{1m}(X + A)$, and the other from the nonlinear term $B_{2m}/(X + A)$. From the above proposition, note that the coefficients B_{1m}, B_{2m} are still functions of Δ , the signed measure of heterogeneity across the agents' background income. However, the relationship is not linear anymore. To see this in more detail, consider the special case of $\rho = 1$ when there are equal numbers of agents in the two groups. Then, the two coefficients are:

$$\begin{aligned} B_{1m} &= \frac{\beta\delta\Delta}{(1 + \Delta\beta)(1 - \Delta\beta)} \{2\beta + \gamma(\theta - \beta)\}, \\ &= \frac{\beta\delta\Delta}{1 - \Delta^2\beta^2} \{2\beta + \gamma(\theta - \beta)\} \\ B_{2m} &= \frac{-2\Delta\beta}{(1 + \Delta\beta)(1 - \Delta\beta)} \\ &= \frac{-2\Delta\beta}{1 - \Delta^2\beta^2} \end{aligned}$$

Given $\theta > \beta$, the coefficient of the additional linear term, B_{1m} , is proportional to $\Delta/(1 - \Delta^2\beta^2)$. The coefficient of the nonlinear term, B_{2m} , is proportional to $-\Delta/(1 - \Delta^2\beta^2)$. Thus, the nonlinear term can be upward or downward sloping in the state X , depending on the sign of B_{2m} . However, the term $1/(X + A)$ is always a decreasing and convex function of the states X , it follows that:

Corollary 4 [*Convexity of the Contingent Claim Demand*]

Assume equal numbers of agents, $\rho = 1$ and $\Delta < 0$. The demand of each m -type agent is a decreasing convex function of X .

Proof

From equation (26),

$$x_{m1}^* = X + \Delta\beta(X + A) - \Delta,$$

or

$$x_{m1}^* + a_m = (X + A)(1 + \Delta\beta).$$

Now, $X + A$ is positive by assumption. Also, we must have $x_{m1}^* + a_m > 0$ and it follows that $(1 + \Delta\beta) > 0$.

The coefficient

$$B_{2m} = \frac{-2\Delta\beta}{(1 + \Delta\beta)(1 - \Delta\beta)}.$$

From its definition in (20), $\beta > 0$ so the numerator of B_{2m} is positive, given $\Delta < 0$. Also the factor $(1 - \Delta\beta)$ in the denominator is positive for the same reason. Since $(1 + \Delta\beta) > 0$, it follows that $B_{2m} > 0$. Finally, since $1/(X + A)$ is a declining, convex function of X , the corollary follows.

Corollary 4 shows how the convexity of the demand for contingent claims is affected by the expected background income of agents. Also, since the β coefficient is a function of the risk aversion parameter γ , it shows that in the CRRA economy, γ controls the degree of convexity of the demand function.

Intuitively, agent m will either long or short a put-like option on the state X . Let us study these conditions in more detail. The condition for the long position is when $B_{2m} > 0$. This is the case when either $\Delta < 0, 1 - \Delta^2\beta^2 > 0$ or $\Delta > 0, 1 - \Delta^2\beta^2 < 0$. If, for example, $\beta \in (0, 1)$, it follows that under the condition $\Delta \in (-1/\beta, 0) \cup (1/\beta, \infty)$, $B_{2m} > 0$. Similarly, under the condition $\Delta \in (-\infty, -1/\beta) \cup (0, 1/\beta)$, $B_{2m} < 0$.¹³

Combining this with the linear demand from B_{1m} , it follows that, in addition to the linear demand from the full resolution case, the effect of the residual risk on the demand of agent is either a downward sloping convex or an upward sloping concave function of the states X .

To summarize, the overall excess demand function of agent m , $x_{m1} - X$, can be characterized as follows:

- When $\Delta > 0$, $x_{m1} - X$ will be an upward sloping linear function of X plus:
 - a downward sloping convex function of X when $\Delta > 1/\beta$;
 - an upward sloping concave function of X when $1/\beta > \Delta > 0$;
- When $\Delta < 0$, $x_{m1} - X$ will be a downward sloping linear function of X plus:
 - a downward sloping convex function of X when $-1/\beta < \Delta < 0$;
 - an upward sloping concave function of X when $\Delta < -1/\beta$;

In Figure 2, we plot the optimal demand for an example. In this example, $A = 0$ and $\rho = 1$. In panel A, the optimal demand x_{m1} is a nonlinear function of states X . We plot the two additional terms (over the demand in the special case) for agent m and agent n in panel B. In this example, $-1/\beta < \Delta < 0$. Thus the two additional effects are downward sloping convex functions of X , while $B_{1n}(X + A) + B_{2q}/(X + A)$ is an upward sloping concave function. A symmetric argument implies that the later corresponds to the case for $1/\beta > \Delta > 0$. The intuition for this example is that agent m is the relatively “poor” agent because $\Delta < 0$. The additional terms (relative to the special case) will induce agent m to demand more in the lower states, while reducing demand in the higher states.

6 Conclusion

There is an extensive literature on background risk, which arises from stochastic cash flows generating non-marketable wealth. Since this risk cannot be directly hedged, it affects the

¹³ $\beta > 1$ case is very similar: There are still four regions of Δ which result in four different patterns of demands. The only differences are the boundaries between the regions.

derived risk aversion of the individual agent. Generally speaking, as documented by several researchers and synthesized by Gollier (2001), in the presence of background risk, agents generally become more risk-averse in their derived utility functions, and thus, behave like a more risk-averse agent would, in the absence of such a risk. This, in turn, influences the demand for insurance.

There has been rather less attention devoted to the pricing of securities and sharing rules in equilibrium, when agents in the economy face background risk. A notable early paper is by FSS, who analyze the equilibrium in such an economy, and derive the portfolio demand of individual agents in this equilibrium. The agents take into account their non-marketable background risk in optimally determining their demand for the marketable assets. Specifically, FSS show that agents with background risk depart from the linear sharing rule that characterizes behavior in complete markets, and may buy or sell non-linear contingent claims such as options.

In this paper, we take the presence of background risk and its influence on risk taking in a different direction. We explore how the prices of assets are determined in equilibrium by the interplay of portfolio demands across agents in the economy, which take into account the background risks they face. If the agents face different background risks, it is reasonable to expect that their portfolio demands will differ: this is the argument first made by FSS. We extend this argument to the multi-period setting and derive the changes in the portfolio demand of different agents as the background risk is revealed over time. To the extent that these changes differ across agents, it establishes a motive for trading, even in the presence of symmetric (full) information across agents.

The equilibrium we obtain turns out to be fairly complex, since portfolio demands depend on the changed derived risk aversion of agents in the presence of background risk, which in turn, depends on the portfolio holdings. We break this circularity by considering special cases of the evolution of background risk, as well as by using some approximations. We confirm these results by numerical computations.

We have thus been able to derive a theory of trading in the presence of full information, without running afoul of the powerful no-trade results of Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) in the context of asymmetric information models. We believe our theory can be extended in several directions to separate the trading in linear (stocks and bonds) versus non-linear (options) claims. Potentially, our theory is testable, if one can quantify the influences of background risks such as human and housing wealth. This could be of interest to researchers in asset pricing, where the focus is mainly on returns, but could also be related to the aspects of trading analyzed in this paper.

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7 Appendix: Derivation of Demand Equations: The General Case

For convenience define:

$$\begin{aligned}\hat{x}_{m1} &\equiv x_{m1} + a_m \\ \hat{x}_{n1} &\equiv x_{n1} + a_n.\end{aligned}$$

Then, it follows that the optimal demand for agent m can thus be written as:

$$\hat{x}_{m1} = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_m - \psi_{m1})]}{E_1[(X + A - \psi_1)^{1-\gamma]}(X + A - \psi_1) + \psi_{m1}},$$

where

$$\begin{aligned}\psi_{m1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}} \\ \psi_{n1} &= \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{n1}} \\ \psi_1 &= \frac{(1 + \gamma)\sigma_\eta^2}{2(1 + \rho)}\left(\frac{\rho}{\hat{x}_{m1}} + \frac{1}{\hat{x}_{n1}}\right) \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}}\left[\frac{1}{1 + \rho}(\rho\hat{x}_{n1} + \hat{x}_{m1})\right] \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}}\left[X + A + \frac{1 - \rho}{1 + \rho}(\hat{x}_{m1} - \hat{x}_{n1})\right] \\ &= \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}}(X + A)(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}),\end{aligned}$$

and

$$\begin{aligned}\Delta_{\hat{x}_{m1}\hat{x}_{n1}} &\equiv \frac{1 - \rho}{1 + \rho} \cdot \frac{\hat{x}_{m1} - \hat{x}_{n1}}{X + A} \\ &= \frac{1 - \rho}{1 + \rho} \cdot \frac{(x_{m1} - x_{n1}) + (a_m - a_n)}{X + A}\end{aligned}$$

Similarly, the optimal demand for agent n is

$$\hat{x}_{n1} = \frac{E_1[(X + A - \psi_1)^{-\gamma}(X + a_n - \psi_{n1})]}{E_1[(X + A - \psi_1)^{1-\gamma]}(X + A - \psi_1) + \psi_{n1}},$$

The required approximations are:

$$\begin{aligned}
(X + A - \psi_1)^{-\gamma} &= \left[X + A - \frac{(1 + \gamma)\sigma_\eta^2}{2} \frac{X + A}{\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right]^{-\gamma} \\
&= (X + A)^{-\gamma} \left[1 - \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right]^{-\gamma} \\
&\approx (X + A)^{-\gamma} \left[1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right],
\end{aligned}$$

where in the last step we use the approximation that $\sigma_\eta^2/(\hat{x}_{m1}\hat{x}_{n1})$ is small.

Similarly, we obtain the approximation:

$$(X + A - \psi_1)^{1-\gamma} \approx (X + A)^{1-\gamma} \left[1 - \frac{(1 - \gamma^2)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right].$$

Thus:

$$\frac{1}{E_1 \left\{ (X + A)^{1-\gamma} \left[1 - \frac{(1-\gamma^2)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right] \right\}} \approx \frac{1}{E_1[(X + A)^{1-\gamma}]} \left\{ 1 + \frac{E_1 \left[\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right]}{E_1[(X + A)^{1-\gamma}]} \right\}$$

Substituting these into the optimal demand function, it follows:

$$\begin{aligned}
\hat{x}_{m1} &\approx E_1 \left[(X + A)^{-\gamma} \left(1 + \frac{\gamma(\gamma + 1)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right) \left(X + a_m - \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}} \right) \right] \\
&\quad \frac{1}{E_1[(X + A)^{1-\gamma}]} \left[1 + \frac{E_1 \left(\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right)}{E_1[(X + A)^{1-\gamma}]} \right] \\
&\quad \left(X + A - \frac{(1 + \gamma)\sigma_\eta^2(X + A)}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \right) + \frac{(1 + \gamma)\sigma_\eta^2}{2\hat{x}_{m1}}.
\end{aligned}$$

Then, under our assumption the terms σ^4/\hat{x}_{m1}^4 , $\sigma^4/\hat{x}_{m1}^3\hat{x}_{n1}$, $\sigma^4/\hat{x}_{m1}^2\hat{x}_{n1}^2 \rightarrow 0$. Thus we have:

$$\hat{x}_{m1} \approx E_1 \left[(X + A)^{-\gamma} \left(X + a_m + \frac{\gamma(1 + \gamma)\sigma_\eta^2(X + a_m)}{2\hat{x}_{m1}\hat{x}_{n1}} (1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) - \frac{(\gamma + 1)\sigma_\eta^2}{2\hat{x}_{m1}} \right) \right] \frac{(X + A)}{E[(X + A)^{1-\gamma}]}$$

$$\begin{aligned}
& \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) + \frac{E_1\left(\frac{(1-\gamma^2)\sigma_\eta^2(X+A)^{1-\gamma}}{2\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right)}{E_1[(X+A)^{1-\gamma}]} \right] + \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}} \\
= & \left\{ \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \right. \\
& \left. + \frac{(1+\gamma)\sigma_\eta^2(X+A)}{2E_1[(X+A)^{1-\gamma}]} \left[E_1\left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right) - E_1\left(\frac{(X+A)^{-\gamma}}{\hat{x}_{m1}}\right) \right] \right\} \\
& \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1\left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right)}{E_1[(X+A)^{1-\gamma}]} \right] + \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}}
\end{aligned}$$

Further, combining terms in the above expression, it follows:

$$\begin{aligned}
\hat{x}_{m1} & \approx \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]}(X+A) \left[E_1\left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right) - E_1\left(\frac{(X+A)^{-\gamma}}{\hat{x}_{m1}}\right) \right] \\
& - \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}\hat{x}_{n1}} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A)(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1\left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right)}{E_1[(X+A)^{1-\gamma}]} \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) + \frac{(1+\gamma)\sigma_\eta^2}{2\hat{x}_{m1}} \\
= & \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X+A) \\
& + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1\left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right) \right. \right. \\
& - E_1\left(\frac{(X+A)^{-\gamma}}{\hat{x}_{m1}}\right) + E_1[(X+A)^{-\gamma}(X+a_m)] \frac{E_1\left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}})\right)}{E_1[(X+A)^{1-\gamma}]} \left. \right] (X+A) \\
& - E_1[(X+A)^{-\gamma}(X+a_m)] \frac{(X+A)}{\hat{x}_{m1}\hat{x}_{n1}}(1 + \Delta_{\hat{x}_{m1}\hat{x}_{n1}}) + \frac{E_1((X+A)^{1-\gamma})}{\hat{x}_{m1}} \left. \right\}
\end{aligned}$$

Finally, the approximate explicit solution is found by substituting $\hat{x}_{m1} = \hat{x}_{m1}^*$, $\hat{x}_{n1} = \hat{x}_{n1}^*$ to obtain

$$\hat{x}_{m1} \approx \hat{x}_{m1}^* + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[(X+A)^{1-\gamma}]} \left\{ \left[E_1\left(\frac{\gamma(X+A)^{-\gamma}(X+a_m)}{\hat{x}_{m1}^*\hat{x}_{n1}^*}(1 + \Delta_{\hat{x}_{m1}^*\hat{x}_{n1}^*})\right) - E_1\left(\frac{(X+A)^{-\gamma}}{\hat{x}_{m1}^*}\right) \right] \right.$$

$$\begin{aligned}
& + E_1 [(X + A)^{-\gamma}(X + a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{\hat{x}_{m1}^* \hat{x}_{n1}^*} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}) \right)}{E_1[(X + A)^{1-\gamma}]} \Big] (X + A) \\
& - E_1[(X + A)^{-\gamma}(X + a_m)] \frac{(X + A)}{\hat{x}_{m1}^* \hat{x}_{n1}^*} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}) + \frac{E_1((X + A)^{1-\gamma})}{\hat{x}_{m1}^*} \Big\} \\
& = \hat{x}_{m1}^* + \frac{(1 + \gamma)\sigma_\eta^2}{2} \left\{ B_{1m}(X + A) - \frac{E_1[(X + A)^{-\gamma}(X + a_m)](X + A)}{E_1((X + A)^{1-\gamma})\hat{x}_{m1}^* \hat{x}_{n1}^*} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}) + \frac{1}{\hat{x}_{m1}^*} \right\} \\
& = \hat{x}_{m1}^* + \frac{(1 + \gamma)\sigma_\eta^2}{2} \left[B_{1m}(X + A) + B_{2m} \frac{1}{(X + A)} \right],
\end{aligned}$$

where

$$\begin{aligned}
B_{1m} & = \frac{1}{E_1[(X + A)^{1-\gamma}]} \left[E_1 \left(\frac{\gamma(X + A)^{-\gamma}(X + a_m)}{\hat{x}_{m1}^* \hat{x}_{n1}^*} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}) \right) - E_1 \left(\frac{(X + A)^{-\gamma}}{\hat{x}_{m1}^*} \right) \right. \\
& \quad \left. + E_1 [(X + A)^{-\gamma}(X + a_m)] \frac{E_1 \left(\frac{(1-\gamma)(X+A)^{1-\gamma}}{\hat{x}_{m1}^* \hat{x}_{n1}^*} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}) \right)}{E_1[(X + A)^{1-\gamma}]} \right], \\
B_{2m} & = \frac{E_1[(X + A)^{1-\gamma}]}{E_1[(X + A)^{-\gamma}(X + a_m)]} - \frac{E_1[(X + A)^{1-\gamma}]}{E_1[(X + A)^{-\gamma}(X + a_n)]} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}).
\end{aligned}$$

Using the expression for x_{m1}^* , x_{n1}^* , we can obtain an explicit expression for $\Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}$:

$$\begin{aligned}
\Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*} & = \frac{1 - \rho (x_{m1}^* + a_m) - (x_{n1}^* + a_n)}{1 + \rho (X + A)} \\
& = \frac{1 - \rho \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma}]}(X + A) - \frac{E_1[(X+A)^{-\gamma}(X+a_n)]}{E_1[(X+A)^{1-\gamma}]}(X + A)}{1 + \rho (X + A)} \\
& = \frac{1 - \rho (a_m - a_n) E_1[(X + A)^{-\gamma}]}{1 + \rho E_1[(X + A)^{1-\gamma}]} \\
& = \frac{(1 - \rho) \Delta E_1[(X + A)^{-\gamma}]}{E_1[(X + A)^{1-\gamma}]} \\
& = (1 - \rho) \Delta \beta.
\end{aligned}$$

We can further simplify the explicit expression for the two coefficients B_{1m} , B_{2m} . Note that:

$$B_{2m} = \frac{E_1[(X + A)^{1-\gamma}]}{E_1[(X + A)^{-\gamma}(X + a_m)]} - \frac{E_1[(X + A)^{1-\gamma}]}{E_1[(X + A)^{-\gamma}(X + a_n)]} (1 + \Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*}),$$

where

$$\Delta_{\hat{x}_{m1}^* \hat{x}_{n1}^*} = \frac{(1 - \rho) \Delta E_1[(X + A)^{-\gamma}]}{E_1[(X + A)^{1-\gamma}]} = (1 - \rho) \Delta \beta.$$

Furthermore:

$$\begin{aligned}
\frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_m)]} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+A+a_m-A)]} \\
&= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{1-\gamma}] + \Delta E_1[(X+A)^{-\gamma}]} \\
&= \frac{1}{1 + \Delta\beta} \\
\frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+a_n)]} &= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{-\gamma}(X+A+a_n-A)]} \\
&= \frac{E_1[(X+A)^{1-\gamma}]}{E_1[(X+A)^{1-\gamma}] - \rho\Delta E_1[(X+A)^{-\gamma}]} \\
&= \frac{1}{1 - \rho\Delta\beta}
\end{aligned}$$

It follows that:

$$\begin{aligned}
B_{2m} &= \frac{1}{1 + \Delta\beta} - \frac{1 + (1 - \rho)\Delta\beta}{1 - \rho\Delta\beta} \\
&= \frac{1 - \rho\Delta\beta - [1 + (1 - \rho)\Delta\beta](1 + \Delta\beta)}{(1 + \Delta\beta)(1 - \rho\Delta\beta)} \\
&= \frac{1 - \rho\Delta\beta - (1 + \Delta\beta) - \Delta\beta(1 - \rho)(1 + \Delta\beta)}{(1 + \Delta\beta)(1 - \rho\Delta\beta)} \\
&= \frac{-\Delta\beta[1 + \rho + 1 - \rho + \Delta\beta - \rho\Delta\beta]}{(1 + \Delta\beta)(1 - \rho\Delta\beta)} \\
&= \frac{-\Delta\beta[2 + (1 - \rho)\Delta\beta]}{(1 + \Delta\beta)(1 - \rho\Delta\beta)}
\end{aligned}$$

Now for B_{2m} , note that:

$$\begin{aligned}
\hat{x}_{m1}^* &= \frac{E_1[(X+A)^{-\gamma}(X+a_m)]}{E_1[(X+A)^{1-\gamma]}(X+A)} \\
&= (1 + \Delta\beta)(X+A), \\
\hat{x}_{n1}^* &= \frac{E_1[(X+A)^{-\gamma}(X+a_n)]}{E_1[(X+A)^{1-\gamma]}(X+A)} \\
&= (1 - \rho\Delta\beta)(X+A).
\end{aligned}$$

Then

$$\hat{x}_{m1}^* \hat{x}_{n1}^* = (1 + \Delta\beta)(1 - \rho\Delta\beta)(X+A)^2.$$

It follows that:

$$\begin{aligned}
B_{1m} &= \frac{1}{E_1[(X+A)^{1-\gamma}]} \left\{ \frac{1+(1-\rho)\Delta\beta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \gamma E_1[(X+A)^{-2-\gamma}(X+A+\Delta)] \right. \\
&\quad \left. - \frac{E_1[(X+A)^{-1-\gamma}]}{1+\Delta\beta} + \frac{1+(1-\rho)\Delta\beta}{(1+\Delta\beta)(1-\rho\Delta\beta)} (1-\gamma) E_1[(X+A)^{-1-\gamma}](1+\Delta\beta) \right\} \\
&= \frac{1+(1-\rho)\Delta\beta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \gamma \left[\frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{1-\gamma}]} + \Delta \frac{E_1[(X+A)^{-2-\gamma}]}{E_1[(X+A)^{1-\gamma}]} \right] \\
&\quad - \frac{1}{1+\Delta\beta} \frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{1-\gamma}]} + \frac{1+(1-\rho)\Delta\beta}{1-\rho\Delta\beta} (1-\gamma) \frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{1-\gamma}]}
\end{aligned}$$

Define

$$\theta \equiv \frac{E_1[(X+A)^{-2-\gamma}]}{E_1[(X+A)^{-1-\gamma}]}.$$

We can rewrite the following:

$$\begin{aligned}
\frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{1-\gamma}]} &= \frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{-\gamma}]} \frac{E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]} \\
&= \beta\delta \\
\frac{E_1[(X+A)^{-2-\gamma}]}{E_1[(X+A)^{1-\gamma}]} &= \frac{E_1[(X+A)^{-2-\gamma}]}{E_1[(X+A)^{-1-\gamma}]} \frac{E_1[(X+A)^{-1-\gamma}]}{E_1[(X+A)^{-\gamma}]} \frac{E_1[(X+A)^{-\gamma}]}{E_1[(X+A)^{1-\gamma}]} \\
&= \beta\delta\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
B_{1m} &= \frac{1+(1-\rho)\Delta\beta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \gamma [\beta\delta + \Delta\beta\delta\theta] \\
&\quad - \frac{1}{1+\Delta\beta} \beta\delta + \frac{1+(1-\rho)\Delta\beta}{1-\rho\Delta\beta} (1-\gamma) \beta\delta \\
&= \frac{\beta\delta}{(1+\Delta\beta)(1-\rho\Delta\beta)} [(1+(1-\rho)\Delta\beta)\gamma(1+\Delta\theta) - \\
&\quad -(1-\rho\Delta\beta) + (1+\Delta\beta)(1+(1-\rho)\Delta\beta)(1-\gamma)] \\
&= \frac{\beta\delta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \{(1+(1-\rho)\Delta\beta)[\gamma(1+\Delta\theta) + (1+\Delta\beta)(1-\gamma)] - (1-\rho\Delta\beta)\} \\
&= \frac{\beta\delta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \{(1+(1-\rho)\Delta\beta)[(1+\Delta\beta) + \gamma\Delta(\theta-\beta)] - (1-\rho\Delta\beta)\} \\
&= \frac{\beta\delta}{(1+\Delta\beta)(1-\rho\Delta\beta)} \{(1+(1-\rho)\Delta\beta)(1+\Delta\beta) - (1-\rho\Delta\beta) \\
&\quad + \gamma(1+(1-\rho)\Delta\beta)\Delta(\theta-\beta)\}
\end{aligned}$$

$$= \frac{\beta\delta\Delta}{(1 + \Delta\beta)(1 - \rho\Delta\beta)} \{\beta(2 + (1 - \rho)\Delta\beta) + \gamma(1 + (1 - \rho)\Delta\beta)(\theta - \beta)\}$$

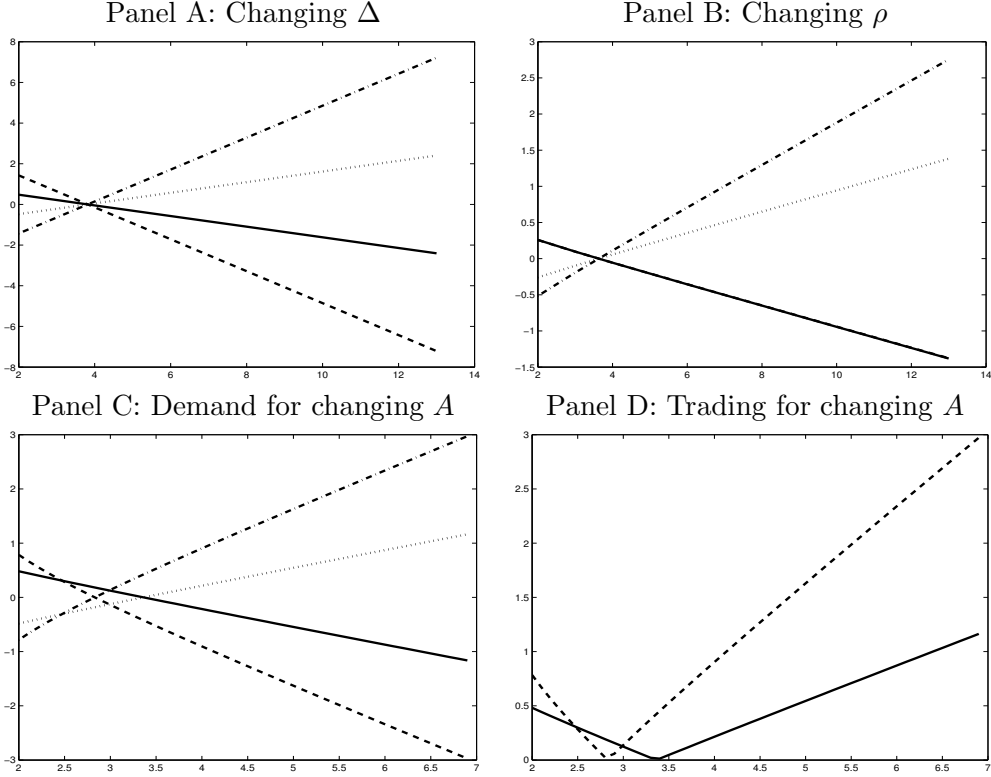


Figure 1: Special Case

This figure shows the comparative static results for the special case analyzed in section (): *Panel A* denotes the optimal excess demand of the two agents when there is a change in Δ . Here, $A = 0$ and $\rho = 1$. The solid line is for $x_{m1}^* - X$ with $\Delta = -1$; The dashed line is for $x_{m1}^* - X$ with $\Delta = -3$; The dotted line is for $x_{n1}^* - X$ with $\Delta = -1$; and the dotted dashed line is for $x_{n1}^* - X$ with $\Delta = -3$.

Panel B denotes the optimal excess demand of the two agents when there is a change in ρ , holding A, Δ fixed. The solid line (and the overlapping dashed line) is for x_{m1}^* ; The dotted line is for $x_{n1}^* - X$ with $\rho = 1$; The dotted dashed line is for $x_{n1}^* - X$ with $\rho = 2$.

Panel C denotes the optimal excess demand of the two agents when there is a change in A , holding $\Delta = -0.5$ and $\rho = 1$. The solid line is for $x_{m1}^* - X$ with $A = 0$; The dashed line is for $x_{m1}^* - X$ with $A = -1$; The dotted line is for $x_{n1}^* - X$ with $A = 0$; The dotted dashed line is for $x_{n1}^* - X$ with $A = -1$.

Panel D denotes the trading by agent m when there is a change in A , holding $\Delta = -0.5$ and $\rho = 1$. The solid line is the trading z_{m1} with $A = 0$; The dashed line is the trading z_{m1} with $A = -1$.

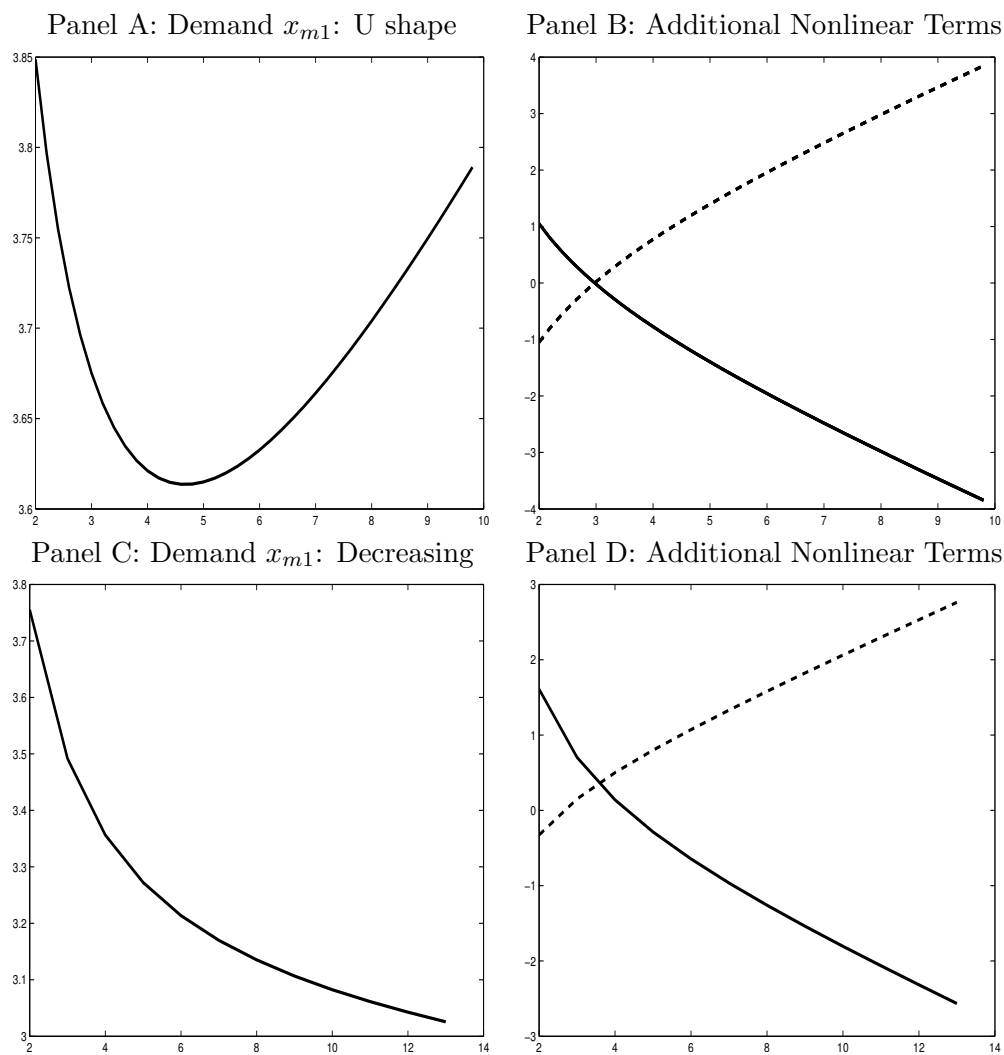


Figure 1: This shows the demands for general case: **Panel A** denotes the optimal demand of agent m . **Panel B** denotes the additional terms of agent m and n . In these two panels, X goes from 2 to 9.8 with 0.2 interval (so totally 40 states). Other parameters are $\gamma = 2, \Delta = -3, \sigma_\eta = 0.5$. **Panel C** denotes the optimal demand of agent m . **Panel D** denotes the additional terms of agent m and n . In these two panels, X goes from 2 to 13 with interval 1 (so totally 12 states). Other parameters are $\gamma = 2, \Delta = -3, \sigma_\eta = 0.5$.