Advanced Derivatives: Course Notes

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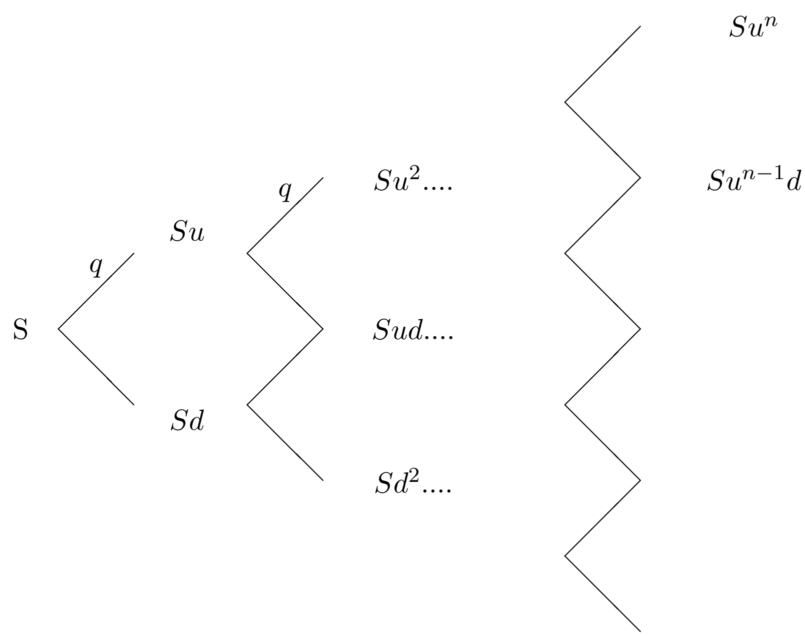
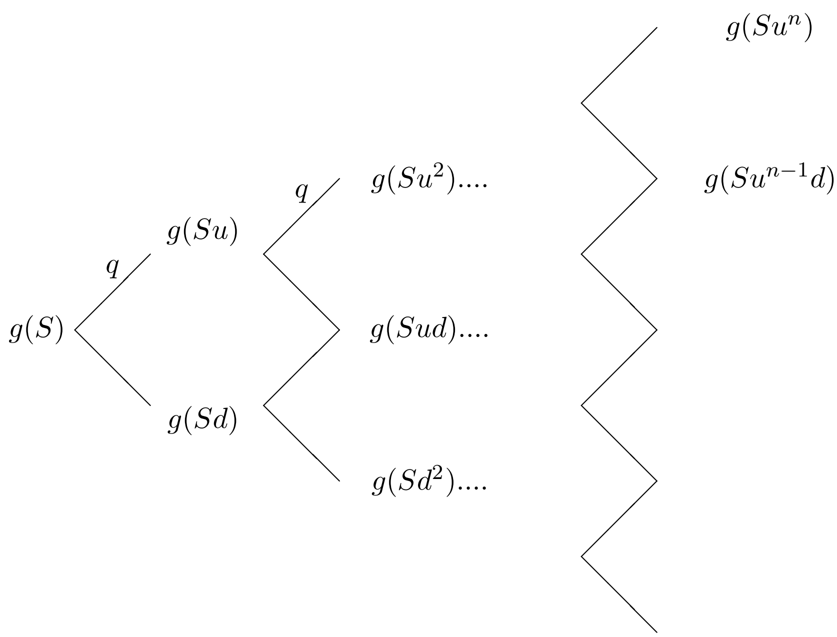
1 The Binomial Model

- Assume that we know that the stock price follows a geometric process with constant proportionate up and down movements, u and d:



where q is the probability of an up move.

- A contingent claim (for example a call or a put option) has a price $g(S)$ which follows the process:



- Define the hedge ratio:

$$\delta_1 = \frac{g(Su) - g(Sd)}{Su - Sd}$$

Lemma 1 *The portfolio of δ_1 stocks and 1 short contingent claim has a riskless payoff at $t = 1$ equal to*

$$\frac{g(Su)d - g(Sd)u}{u - d}$$

Proposition 1.1 *Suppose the price of a stock and a contingent claim follow the processes above, then the no-arbitrage price of the contingent claim is*

$$g(S) = \frac{[pg(Su) + (1-p)g(Sd)]}{R}$$

where

$$p = \frac{R-d}{u-d}$$

and R is 1+ risk-free rate.

Corollary 1 *Suppose the price of a stock and a contingent claim follow the processes above, then the no-arbitrage price of the contingent claim is*

$$g(S) = \frac{[p^n g(Su^n) + p^{n-1}(1-p)g(Su^{n-1}d)n + ... + p^{n-r}(1-p)^r g(Su^{n-r}d^r)\frac{n!}{r!(n-r)!} + ...]}{R^n}$$

where

$$n! = n(n-1)(n-2)...(2)(1),$$

$$\frac{n!}{r!(n-r)!}$$

is the number of paths leading to node r, and

r is the number of down moves of the process.

Example 1: A Call Option

A call option with maturity T and strike price K has a payoff $\max[S_T - K, 0]$ at time T.

$$R^n g(S_t) = \frac{[p^n(S_t u^n - K) + p^{n-1}(1-p)n(S_t u^{n-1}d - K) + ... + p^{n-r}(1-p)^r(S_t u^{n-r}d^r - K)\frac{n!}{r!(n-r)!}}{R^n}$$

$$\begin{aligned}\frac{R^n g(S_t)}{S_t} &= p^n u^n + p^{n-1}(1-p)u^{n-1}dn + ... + p^{n-r}(1-p)^r u^{n-r}d^r \frac{n!}{r!(n-r)!} \\ &- k\left[p^n + p^{n-1}(1-p)n + ... + p^{n-r}(1-p)^r \frac{n!}{r!(n-r)!}\right]\end{aligned}$$

$$\begin{aligned}
\frac{R^n g(S_t)}{S_t} &= \sum_{i=0}^{r} p^{n-i}(1-p)^i u^{n-i} d^i \frac{n!}{i!(n-i)!} \\
&- k \sum_{i=0}^{r} p^{n-i}(1-p)^i \frac{n!}{i!(n-i)!}
\end{aligned}$$

The Black-Scholes Formula

Define $B_{t,T} = \frac{1}{R^n}$ and

$$P(i) = p^{n-i}(1-p)^i \frac{n!}{i!(n-i)!}$$

$$S_{T,i} = S_t u^{n-i} d^i$$

Then

$$g(S_t) = B_{t,T} \left[\sum_{i=0}^{r} S_{T,i} P(i) - K \sum_{i=0}^{r} P(i) \right]$$

and in the limit as $n \to \infty$, we have

$$g(S_t) = B_{t,T} \int_K^{\infty} S_T f(S_T) dS_T - K B_{t,T} \int_K^{\infty} f(S_T) dS_T$$

where $f(S_T)$ is the distribution of S_T under the risk-neutral probabilities, P.

Proof of Proposition 1.1

From lemma 1 the payoff on the hedge portfolio of δ_1 stocks and one short option is

$$\frac{g(Su)d - g(Sd)u}{u-d}$$

Since the payoff is risk free its value must be

$$\delta_1 S - g(S) = \frac{g(Su)d - g(Sd)u}{R(u-d)}$$

Hence

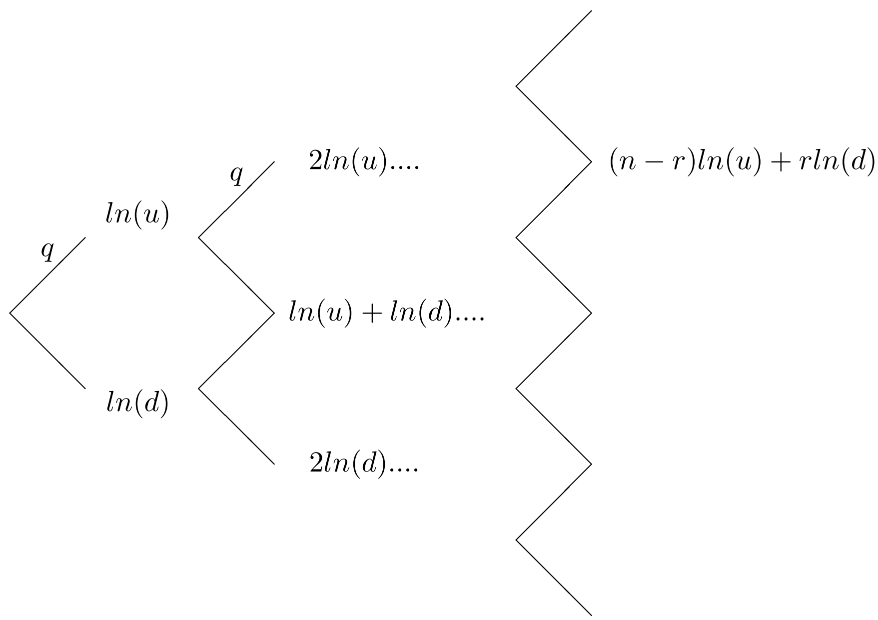
$$\begin{aligned}
g(S) &= \frac{g(Su) - g(Sd)}{u-d} - \frac{g(Su)d - g(Sd)u}{R(u-d)} \\
&= \frac{Rg(Su) - Rg(Sd) - g(Su)d + g(Sd)u}{R(u-d)} \\
&= \frac{1}{R}\left[g(Su)\left(\frac{R-d}{u-d}\right) - g(Sd)\left(\frac{u-R}{u-d}\right)\right] \\
&= \frac{1}{R}\left[g(Su)p - g(Sd)(1-p)\right]
\end{aligned}$$

where

$$p = \frac{R-d}{u-d}$$

$$1-p = \frac{u-R}{u-d}$$

Assume the asset price follows the log-binomial process:



If x_n follows the above process, the logarithm of x_n has a mean:

$$n[q \ln(u) + (1 - q) \ln(d)] = \mu(T - t)$$

a variance:

$$n[q(1 - q)(\ln(u) - \ln(d))^2] = \sigma^2(T - t).$$

Lemma 2 *Assume that two lognormally distributed stocks have the same volatility, σ, and have mean, μ_j, $j = 1, 2$. Then the lognormal distributions can be approximated with log-binomial distributions with $(u, d, q_1 = 0.5)$ and (u, d, q_2) for large n.*

Proposition 1.2 *Consider two stocks with prices $S_1 = S_2$ and volatility σ and a derivative with exercise price K. Let the derivative prices be $g(S_1), g(S_2)$. Then, regardless of the drifts μ_1, μ_2*

$$g(S_1) = g(S_2)$$

Proof

We consider the hedge ratio for each option in state u at time 1. First we have

$$g(S_1u^2) = g(S_2u^2)$$

$$g(S_1ud) = g(S_2ud)$$

The hedge ratio for option 1 is

$$\delta_{1,1,u} = \frac{g(S_1u^2) - g(S_1ud)}{S_1(u-d)} = \delta_{1,2,u}$$

Now consider a portfolio of two stocks and two options costing

$$\delta_{1,1,u}S_1u - g(S_1u) - [\delta_{1,2,u}S_2u - g(S_2u)] = -g(S_1u) + g(S_2u)$$

This portfolio provides a risk-free return equal to

$$\delta_{1,1,u}S_1u^2 - g(S_1u^2) - [\delta_{1,2,u}S_2u^2 - g(S_2u^2)] = 0$$

It follows that

$$g(S_1u) = g(S_2u)$$

By a similar argument

$$g(S_1d) = g(S_2d)$$

and also

$$g(S_1) = g(S_2)$$

Proposition 1.3 *(Corhay and Stapleton) Consider two stocks with the same volatility* σ*. Assume that* $S_{1,t}$ *and* $S_{2,T}$ *follow the diffusion processes*

$$\frac{dS_{1,t}}{S_{1,t}} = \mu_1 dt + \sigma dz$$

and

$$\frac{dS_{2,t}}{S_{2,t}} = \mu_2 dt + \sigma dz$$

where μ_1 *and* μ_2 *are the drift parameters for assets 1 and 2.*

Assume that there are two derivatives securities with the same contract specifications, and exercise prices K_1 *and* K_2 *such that*

$$\frac{K_1}{S_{1,0}} = \frac{K_2}{S_{2,0}}$$

i.e. the strike price relative to the stock price at time 0 is the same. Let the price at time 0 of the derivative on asset 1 be $g_1(S_{1,0})$ *and on asset 2 be* $g_2(S_{2,0})$*. Then in the absence of arbitrage*

$$\frac{g_1(S_{1,0})}{S_{1,0}} = \frac{g_2(S_{2,0})}{S_{2,0}}.$$

2 The Black-Scholes and Black Models

Given the Mean-irrelevance Theorem, an option can be valued by valuing an equivalent option on a 'risk-neutral' stock, with volatility σ. We need the following:

Lemma 3 *The value of a call option on a risk-neutral stock, i.e. a stock, paying no dividends, that has a price*

$$S_t = B_{t,T}E(S_T),$$

is

$$C_t = B_{t,T}E[max(S_T - K, 0)].$$

Hence, if S_T is lognormal, a call option on a risk-neutral stock has a value:

$$C_t = B_{t,T}F[g(S_T)],$$

where $F(.)$ denotes 'forward price of', and

$$g(S_T) = max(S_T - K, 0)$$

$$\begin{aligned}
\frac{F[g(S_T)]}{S_t} &= E\left[max\left(\frac{S_T}{S_t} - k, 0\right)\right] \\
&= \int_k^\infty \left(\frac{S_T}{S_t} - k\right) f\left(\frac{S_T}{S_t}\right) d\left(\frac{S_T}{S_t}\right) \\
&= \int_{\ln(k)}^\infty (e^z - k)\, f(z) d(z)
\end{aligned}$$

where $k = \frac{K}{S_t}$ and $z = \ln\left(\frac{S_T}{S_t}\right)$

Lemma 4 *If $f(y)$ is normal with mean μ and standard deviation $\hat{\sigma}$ then*

1.

$$\int_a^\infty f(y)d(y) = N\left(\frac{\mu - a}{\hat{\sigma}}\right)$$

and

2.

$$\int_a^\infty e^y f(y)d(y) = N\left(\frac{\mu - a}{\hat{\sigma}} + \hat{\sigma}\right) e^{\mu + \frac{1}{2}\hat{\sigma}^2}$$

with

3.

$$E(e^y) = e^{\mu + \frac{1}{2}\hat{\sigma}^2}$$

We have in this case

$$\mu = E\left[\ln\left(\frac{S_T}{S_t}\right)\right]$$

and

$$\hat{\sigma}^2 = \sigma^2(T - t)$$

From risk neutrality $E(S_T) = F$ and hence, using the Lemma 4, 3

$$E\left[\frac{S_T}{S_t}\right] = \frac{F}{S_t} = e^{\mu + \frac{1}{2}\sigma^2(T-t)}$$

and it follows that

$$\mu = \ln\left(\frac{F}{S_t}\right) - \frac{1}{2}\sigma^2(T - t)$$

Hence, choosing $a = \ln(k) = \ln\left(\frac{K}{S_t}\right)$,

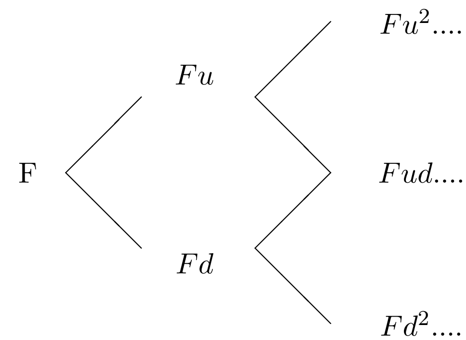
$$\int_a^\infty f(z)d(z) = N\left(\frac{\mu - a}{\hat{\sigma}}\right) = N\left[\frac{\ln\left(\frac{F}{S_t}\right) - \frac{1}{2}\sigma^2(T - t) - \ln\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T - t}}\right]$$

$$\int_a^{\infty} e^z f(z)d(z) = N\left(\frac{\mu - a}{\hat{\sigma}} + \hat{\sigma}\right) = N\left[\frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2(T-t) + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right]\frac{F}{S_t}$$

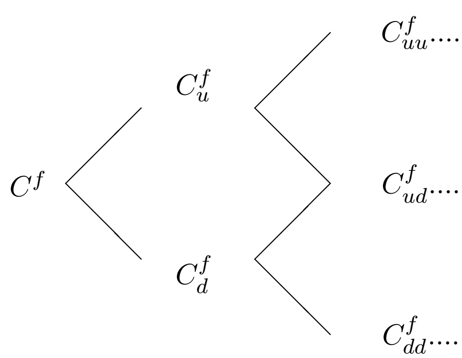
and hence

$$\frac{F[g(S_T)]}{S_t} = \frac{F}{S_t}N\left[\frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right] - KN\left[\frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right]$$

A Proof of the Black Model (Forward Version) Assume the following the process for the forward price of the asset:



and for the forward price of a contingent claim:



Proposition 2.1 *Suppose the forward prices of a stock and a contingent claim follow the processes above, then the no-arbitrage forward price of the contingent claim is*

$$C^f = [p'C_u^f + (1-p')C_d^f]$$

where

$$p' = \frac{1-d}{u-d}$$

If an option pays $max(S_T - K, 0)$, after n sub-periods, then its forward price is given by

$$C^f = E\left[max(S_T - K, 0)\right]$$

where the expectation $E(.)$ is taken over the probabilities p'. Also,

$$\begin{aligned}
\frac{C^f}{F} &= E\left[max(\frac{S_T}{F} - k', 0)\right] \\
&= p^n u^n + p^{n-1}(1-p)u^{n-1}dn + ... + p^{n-r}(1-p)^r u^{n-r} d^r \frac{n!}{r!(n-r)!} \\
&- k'\left[p^n + p^{n-1}(1-p)n + ... + p^{n-r}(1-p)^r \frac{n!}{r!(n-r)!}\right] \\
&= \int_{\ln(k')}^{\infty} (e^z - k')f(z)dz
\end{aligned}$$

where $k' = \frac{K}{F}$ and $z = \ln\left(\frac{S_T}{F}\right)$.

To value the option we again take the case of a 'risk-neutral' stock, where $E(S_T) = F$. Let

$$\mu = E\left[\ln\left(\frac{S_T}{F}\right)\right]$$

We then have

$$E\left[\frac{S_T}{F}\right] = \frac{E[S_T]}{F} = 1 = e^{\mu + \frac{1}{2}\sigma^2}$$

It follows that

$$\mu = -\frac{1}{2}\sigma^2(T-t)$$

Hence, choosing $a = \ln(k') = \ln\left(\frac{K}{F}\right)$,

$$\int_a^\infty f(z)d(z) = N\left(\frac{\mu - a}{\hat{\sigma}}\right) = N\left[\frac{-\frac{1}{2}\sigma^2(T-t) + \ln\left(\frac{F}{K}\right)}{\sigma\sqrt{T-t}}\right]$$

$$\int_a^\infty e^z f(z)d(z) = N\left(\frac{\mu - a}{\hat{\sigma}} + \hat{\sigma}\right) = N\left[\frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2(T-t) + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right]$$

and hence

$$\frac{C^f}{F} = N\left[\frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right] - \frac{K}{F}N\left[\frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right]$$

and

$$\begin{aligned} \frac{C^f}{F} &= N(d_1) - \frac{K}{F}N(d_2) \\ C^f &= FN(d_1) - KN(d_2) \end{aligned}$$

The spot price of the option is

$$C = B_{t,T}FN(d_1) - B_{t,T}KN(d_2)$$

Example 1: A Call Option, Non-Dividend Paying Stock

$$S_t = B_{t,T}F$$

implies

$$C = S_tN(d_1) - B_{t,T}KN(d_2)$$

as in Black-Scholes. Note proof has not assumed constant (non-stochastic interest rates.

Example 2: A Stock Paying Dividends

Assume that dividends are paid at a continuous rate d. The forward price of the stock is

$$F = F_{t,T} = S_te^{(r-d)(T-t)}$$

$$C = B_{t,T}S_te^{(r-d)(T-t)}N(d_1) - B_{t,T}KN(d_2)$$

$$C = S_te^{(-d)(T-t)}N(d_1) - B_{t,T}KN(d_2)$$

since

$$B_{t,T} = s^{-r(T-t)}$$

3 Hedge Ratios in the Black-Scholes and Black Models

We need to know the following sensitivities:

1. The call delta
$$\Delta_c = \frac{\partial C_t}{\partial S_t}$$
2. The put delta
$$\Delta_p = \frac{\partial P_t}{\partial S_t}$$
3. The put and the call gamma
$$\Gamma = \frac{\partial}{\partial S_t}\left[\frac{\partial C_t}{\partial S_t}\right] = \frac{\partial}{\partial S_t}\left[\frac{\partial P_t}{\partial S_t}\right]$$
4. The vega of a call or put option:
$$V = \frac{\partial C_t}{\partial \sigma}$$
5. The theta of a call or put option:
$$\Theta_c = \frac{\partial C_t}{\partial t}$$

The Black-Scholes Model: No Dividends

$$C = S_t N(d_1) - B_{t,T} K N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{F}{K}\right) - \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}.$$

Lemma 5

$$n(d_2)\frac{K}{F} = n(d_1)$$

Lemma 6 (Differential Calculus) *1. Chain rule*

$$\frac{\partial f(g(x))}{\partial x} = g'(x) f'[(g(x)]$$

2. Product rule

$$\frac{\partial[f(x)(g(x)]}{\partial x} = f'(x)g(x) + g'(x)f(x)]$$

Proposition 3.1 *In the Black-Scholes model, the delta of a call option is given by*

$$\Delta_c = \frac{\partial C_t}{\partial S_t} = N(d_1)$$

Proof

Using lemma 7,

$$\frac{\partial C_t}{\partial S_t} = N(d_1) + S_t N'(d_1)\frac{\partial d_1}{\partial S_t} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S_t}$$

Then, note that $d_2 = d_1 - \frac{\sigma\sqrt{T-t}}{2}$ implies $\frac{\partial d_2}{\partial S_t} = \frac{\partial d_1}{\partial S_t}$. Hence,

$$\frac{\partial C_t}{\partial S_t} = N(d_1) + \frac{\partial d_1}{\partial S_t}\left[S_t N'(d_1) - Ke^{-r(T-t)}N'(d_2)\right]$$

and using lemma 5,

$$\frac{\partial C_t}{\partial S_t} = N(d_1) + \frac{\partial d_1}{\partial S_t}\left[S_t N'(d_2)\frac{K}{F} - Ke^{-r(T-t)}N'(d_2)\right]$$

From forward parity

$$F = S_t e^{r(T-t)}$$

and hence

$$\frac{\partial C_t}{\partial S_t} = N(d_1)$$

Corollary 2 *(Put Option Delta) From put-call parity*

$$C_t - P_t = S_t - KB_{t,T}$$

Hence,

$$\frac{\partial C_t}{\partial S_t} - \frac{\partial P_t}{\partial S_t} = 1$$

$$\Delta_p = \frac{\partial P_t}{\partial S_t} = N(d_1) - 1$$

Corollary 3 *(Put, Call Gamma)*

$$\begin{aligned} \frac{\partial}{\partial S_t}\left[\frac{\partial C_t}{\partial S_t}\right] &= \frac{\partial}{\partial S_t}\left[\frac{\partial P_t}{\partial S_t}\right] \\ &= N'(d_1)\frac{\partial d_1}{\partial S_t} \\ &= n(d_1)\frac{1}{S_t\sigma\sqrt{T-t}} \end{aligned}$$

Using a similar method, it is possible to establish

1. The vega of a call option

$$\frac{\partial C_t}{\partial \sigma} = S_t\sqrt{T-t}N'(d_1)$$

and using put-call parity,

$$\frac{\partial P_t}{\partial \sigma} = \frac{\partial C_t}{\partial \sigma}$$

2. The theta of call option

$$\frac{\partial C_t}{\partial t} = \frac{S_t N'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

and using put-call parity

$$\frac{\partial P_t}{\partial t} = \frac{\partial C_t}{\partial t} + rKe^{-r(T-t)}$$

Hedge Ratios: Options on Dividend Paying Stocks

For a stock which pays a continuous dividend at a rate d,

$$C_t = S_t e^{-d(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{ln(S_t/K) + (r - d + \sigma^2)(T-t)/2}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

since, from forward parity,

$$F = S_t e^{(r-d)(T-t)}$$

and hence

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - d + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

It follows that the call delta is

$$\Delta_c = \frac{\partial C_t}{\partial S_t} = e^{-\delta(T-t)} N(d_1)$$

Also, for foreign exchange options

$$\Delta_c = \frac{\partial C_t}{\partial S_t} = e^{-r_f(T-t)} N(d_1)$$

where r_f is the foreign risk-free rate of interest.

Hedge Ratios: Futures-Style Options

Let H be the futures price of the asset at time t, for time T delivery. The Black formula gives a (futures) call value:

$$C^h = HN(d_1) - KN(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{H}{K}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Note that the Black formula will hold if the underlying futures price follows a geometric brownian motion. [The proof is the same as the forward proof above.] Then we have:

Proposition 3.2 *(Call Delta: Futures-Style Options)*

Assume that the option is traded on a marked-to-market basis, then the futures price of the call is given by

$$C^h = HN(d_1) - KN(d_2)$$

and the call delta (in terms of futures positions) is

$$\Delta_c = \frac{\partial C^h}{\partial H} = N(d_1)$$

Proof

For the price C^h, see Satchell, Stapleton and Subrahmanyam (1997). For the hedge ratio, a reworking of Lemma 5 yields in this case

$$\left(\frac{K}{H}\right) n(d_2) = n(d_1)$$

Using this and the same steps as in the proof of proposition 3.1 we get the result.

Corollary 4 *(Libor Futures Options)*

Assuming these are European-style, marked-to-market options, then a put on the futures price has a value

$$P_t = [(1 - H_{t,T})N(d_1) - (1 - K)N(d_2)]$$

where

$$d_1 = \frac{\ln\left(\frac{1-H}{1-K}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where H is the futures price and K is the strike price. The delta hedge ratio, in terms of the underlying Libor futures contract is

$$\frac{\partial P^h}{\partial(1-H)} = N(d_1)$$

$$\frac{\partial P^h}{\partial(H)} = -N(d_1)$$

Proof

The futures price of the option can be established for Libor options by assuming that the futures *rate* follows a lognormal diffusion process (limit of the geometric binomial process as $n \to \infty$). The hedge ratio can be established by reworking Lemma 5 to obtain

$$\left(\frac{1-K}{1-H}\right) n(d_2) = n(d_1)$$

3.1 Hull's Treatment of 'Futures Options'

There are three issues to consider here:

1. **Options on futures.** These are options to enter a futures contract, at a fixed futures price K. However, since in all cases, the maturity of the option is the same as the maturity of the underlying futures, these options have the same payoff as options on spot prices, if exercised at maturity. The main difference is in the valuation of the American-style, early exercise feature (CME v PHILX).

2. **Futures-style options**. On many exchanges (ex. LIFFE) options are traded on a marked-to-market basis, just like the underlying futures. Hull does not deal with the valuation of these options.

3. **Hedging with futures** Any option could be hedged with futures (not necessarily options on futures.

3.2 A Digression on Futures v Forwards

Hull's treatment assumes that there is no significant difference between futures prices and forward prices. [Hull uses the same symbol F for the futures price and the forward price of an asset.] However, this is not true in the case interest rate contracts (especially long-term contracts). Also in the case of options, small differences are magnified.

A long $(x = 0)$ [*short* $(x = 1)$] futures contract made at time t, with maturity T, to buy [sell] an asset at a price $H_{t,T}$ has a payoff profile:

$$(-1)^x \left[H_{t+1,T} - H_{t,T}\right] (-1)^x \left[H_{t+2,T} - H_{t+1,T}\right] \cdots (-1)^x \left[H_{T,T} - H_{T-1,T}\right]$$

On the other hand, a forward contract pays $(-1)^x \left[S_T - F_{t,T}\right]$ at time T.

Pricing

Assume no dividends (up to contract maturity)

$$F_{t,T} = S_t / B_{t,T} = S_t e^{r(T-t)}$$

$$H_{t,T} = F_{t,T} + cov$$

Hull assumes $F_{t,T} = H_{t,T}$ [For hedging this is OK, since $\Delta F_{t,T} \approx \Delta H_{t,T}$]
Under risk neutrality:

$$H_{t,T} = E_t(S_T)$$

and, if interest rates are non-stochastic,

$$F_{t,T} = E_t(S_T)$$

also. However, in general there is a bias due to the covariance term.

Hedging

Forward pays

$$F_{t+1,T} - F_{t,T}$$

at T, which is worth

$$(F_{t+1,T} - F_{t,T})B_{t+1,T}$$

at $t+1$. Futures pays

$$H_{t+1,T} - H_{t,T}$$

at $t+1$, hence the hedge ratios for options, in terms of forwards and futures, are quite different to one another.

Hull does not value 'futures-style' options. His formula for the spot price of a futures option assumes zero covariance between interest rates and the aset price. What if there is a significant correlation (bond options, LIBOR futures options)?

If the forward price of the asset follows a GBM, then the Black model holds, with forward price in the formula. Then the spot price of the option is

$$C = B_{t,T}FN(d_1) - B_{t,T}KN(d_2)$$

But from SSS (1997) this requires $g_{t,T}$ to be lognormal, if the pricing kernel is lognormal.

Conclusion. If Black model holds for futures-style options, it is not likely to hold for spot-style (because of stochastic discounting). It should be established using a forward hedging argument, not a futures hedging argument as in Hull.

4 Approximating Diffusion Processes

Definitions

1. A lognormal diffusion process (geometric Brownian motion) for S_t:

$$dS_t = \mu S_t dt + \sigma S_t dz$$

or

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz$$

In discrete form:

$$\frac{S_{t+1} - S_t}{S_t} = m\mu + \sqrt{m}\sigma\epsilon_{t+1}$$

where m is the length of the time period (in years), and $\epsilon \sim N(0,1)$
Also, we can write

$$d\ln(S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz$$

$$\ln(S_{t+1}) - \ln(S_t) = (\mu - \frac{1}{2}\sigma^2)m + \sigma\sqrt{m}\epsilon_{t+1}$$

2. A constant elasticity of variance (CEV) process for S_t:

$$dS_t = \mu S_t dt + \sigma S_t^\gamma dz$$

(If $\gamma = 1$, lognormal diffusion. If $\gamma = 0$, S_t is normal.) In discrete form:

$$S_{t+1} - S_t = m\mu S_t + \sqrt{m}\sigma S_t^\gamma \epsilon_{t+1}$$

$$var_t(S_{t+1} - S_t) = m\sigma^2 S_t^{2\gamma}$$

$$\frac{\partial var_t(S_{t+1} - S_t)}{\partial S_t} = 2m\gamma\sigma^2 S_t^{2\gamma-1}$$

The elasticity of variance is

$$\frac{\partial var_t(S_{t+1} - S_t)}{\partial S_t}\frac{S_t}{var_t(S_{t+1} - S_t)} = 2\gamma$$

Example: $\gamma = 0.5$, $\surd$ process of Cox, Ingersoll and Ross (1985)

3. A generalized CEV process:

$$dS_t = \mu(S_t, t)S_t dt + \sigma(t)S_t^{\gamma} dz$$

If $\gamma = 0$, $\sigma(t) = \sigma$, and $\mu(S_t, t)S_t = \beta(\alpha - S_t)$

$$dS_t = \beta(\alpha - S_t)dt + \sigma dz$$

This is the Ornstein-Uhlenbeck process, as used in Vasicek (1976) model.

Approximation methods: Lognormal Diffusions

Assume we want to approximate the process

$$dS_t = \mu S_t dt + \sigma S_t dz$$

with a multiplicative binomial with constant u and d movements. From lecture 1, the approximated mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$ are given by:

$$\hat{\mu}T = n[q\ln(u) + (1-q)\ln(d)] \tag{1}$$

$$\hat{\sigma}^2 T = n[q(1-q)(\ln(u) - \ln(d))^2] \tag{2}$$

We need to choose u, d, q so that

$$\hat{\mu}T \to \mu T, \hat{\sigma}T \to \sigma T, n \to \infty$$

1. **The Cox-Rubinstein Solution**

Choose the restriction $ud = 1$, then if $u = e^{\sigma\sqrt{\frac{T}{n}}}$, and

$$q = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right]$$

we have $\hat{\mu} = \mu$. Also, $\hat{\sigma} \to \sigma$, for $n \to \infty$. To prove this note that $ud = 1$ implies $d = e^{-\sigma\sqrt{\frac{T}{n}}}$ and $\ln(d) = -\ln(u)$, and substitution in (1) and (2) gives the result, since $q \to \frac{1}{2}$ as $n \to \infty$.

2. **The Hull-White Solution**

   Choose $q = 0.5$, then the solution to equations (1) and (2), with μ and σ substituted for $\hat{\mu}$ and $\hat{\sigma}$ is

$$\ln(u) = \sqrt{\frac{\sigma^2 T}{n}} + \frac{\mu T}{n}$$

3. **The HSS Solution**

   Suppose we are given a set of expected prices $E_0(S_t)$ for each t, as well as the volatility σ. HSS first construct a process for $x_t = \frac{S_t}{E_0(S_t)}$, where $E_0(x_t) = 1$. To do this, choose $q = \frac{1}{2}$, $u = 2 - d$, and

$$d = \frac{2}{e^{2\sigma\sqrt{\frac{T}{n}}} + 1}.$$

   Then we have $E_0(x_t) = 1$ and $\hat{\sigma} = \sigma$.

Recombining Trees: The Nelson-Ramaswamy Method

[Note: NR use the notation σ for the standard deviation in a Brownian motion, rather than the conventional standard deviation of the logarithm in a geometric Brownian motion. In this section we will use σ' for the NR σ to distinguish it from the volatility (annualised standard deviation of the logarithm), σ.]

NR consider the general process:

$$dy_t = \mu'(y,t) + \sigma'(y,t)dw_t$$

where, for example,

$$\begin{aligned} \mu'(y,t) &= \mu(S_t,t)S_t \\ \sigma'(y,t) &= \sigma(t)S_t^\gamma \end{aligned}$$

If the volatility of the process changes over time, the binomial tree approximation may not combine:

Example 1 GCEV process with $\gamma = 1, \mu(S_t,t) = \mu$

$$dS_t = \mu S_t dt + \sigma(t) S_t dz$$

Lemma 7 *The CEV process approximation is recombining if and only if $\gamma = 0, 1$.*

1. *μ is irrelevant to the recombination issue, so take $\mu = 0$. Recombination requires*
$$S_0 + \sigma S_0^\gamma - \sigma(S_0 + \sigma S_0^\gamma)^\gamma = S_0 - \sigma S_0^\gamma + \sigma(S_0 - \sigma S_0^\gamma)^\gamma$$
If $\gamma = 0$:
$$S_0 + \sigma - \sigma = S_0 - \sigma + \sigma$$
If $\gamma = 1$:
$$S_0 + \sigma S_0 - \sigma(S_0 + \sigma S_0) = S_0 - \sigma S_0 + \sigma(S_0 - \sigma S_0)$$

2. *Take the case where $S_0 = 1$*
$$S_{2,u,d} = 1 + \sigma - \sigma(1+\sigma)^\gamma \neq 1 - \sigma + \sigma(1-\sigma)^\gamma$$

NR split the period $[0, T]$ into n sub-periods of length $h = \frac{T}{n}$. After k sub-periods, y_{hk} goes to $y^+(hk, y_{hk})$ with probability q and to $y^-(hk, y_{hk})$, with probability $(1-q)$. The annualised drift and variance of the process are given by

$$\begin{aligned} h\mu'_h(y,t) &= q[y^+ - y] + (1-q)[[y^- - y] \\ h\sigma'^2(y,t) &= q[y^+ - y]^2 + (1-q)[[y^- - y]^2 \end{aligned}$$

(NR eq 11-12).

1. NR first construct a non-recombining tree. In this tree y goes to $y^+ = y + \sqrt{h}\sigma'(y,t)$ with probability $q = \frac{1}{2} + \sqrt{h}\frac{\mu'(y,t)}{2\sigma'(y,t)}$, and to $y^+ = y - \sqrt{h}\sigma'(y,t)$ with probability $(1-q)$.

2. NR then define a transformation of the process, such that the binomial tree recombines. They choose [NR (25)]
$$x(y,t) = \int^y \frac{dz}{\sigma'(z,t)}$$

in discrete form,

$$x(y,t) = \sum_{1}^{t} \frac{\Delta y_\tau}{\sigma'(y,\tau)} = \frac{\Delta y_1}{\sigma'(y,1)} + \frac{\Delta y_2}{\sigma'(y,2)} + ... + \frac{\Delta y_t}{\sigma'(y,t)}$$

3. NR then define a reverse transformation:

$$y[x(y,t)] : x(y,t) \longrightarrow x(y,t)\sigma'(y,t)$$

Proposition 4.1 *(Nelson and Ramaswamy)*

Suppose y_t is given by the non-recombining tree [NR(21-23)], the transformed process defined by [NR (25-26)] is a simple tree. If we choose the probability of an up-move to match the conditional mean by making

$$q = \frac{h\mu' + y(x,t) - y^-(x,t)}{y^+(x,t) - y^-(x,t)}$$

Then $\hat{\mu}' \to \mu'$ and $\hat{\sigma}' \to \sigma'$, as $n \to \infty$.

Proof

By construction the mean is exact, since

$$q(y^+ - y^-) = h\mu' + y - y^-$$

implies that

$$qy^+ + (1-q)y^- = y + h\mu',$$

i.e. $\hat{\mu}' = \mu'$.

The conditional variance is exact if $q = 0.5$. Also $q \to 0.5$ as $n \to \infty$

5 Multivariate Processes: The HSS Method

Motivation

- For many problems we need to approximate multiple-variable diffusion processes
- It may be reasonable to assume that prices (or rates) follow lognormal diffusions
- From NR if $\ln(X_t) = x_t$ is given by

  $$dx_t = \mu(x_t)dt + \sigma(t)dz$$

  we can build a 'simple' tree for x_t and choose the probability of an up-move

  $$q_{t-1} = \frac{\mu(x_{t-1}) + x_{t-1} - x_t^-}{x_t^+ - x_t^-} \tag{3}$$

HSS assumptions

- X_i is lognormal for all dates t_i, with given mean $E(X_i)$.
- For dates t_i, we are given the local volatilities $\sigma_{i-1,i}$, and the unconditional volatilities $\sigma_{0,i}$.
- Approximate with a binomial process with n_i sub-periods.
- Add a second (or more) variable Y_i, where (X_i, Y_i) are joint log-normal, correlated variables.

Relation to NR: One-Variable Case

Assume an Ornstein-Uhlenbeck process for x_t:

$$dx_t = \kappa(a - x_t)dt + \sigma dz.$$

In discrete form

$$x_i - x_{i-1} = k(a - x_{i-1}) + \sigma\varepsilon_t,$$

$$x_i = ka + b' x_{1-k} + \varepsilon_i' \tag{4}$$

and

$$var(x_i) = (1-k)^2 var(x_{i-1}) + var(\varepsilon_i')$$

In annualised form

$$t_i^2 \sigma_{0,t_i}^2 = (1-k)^2 t_i \sigma_{0,t_i}^2 + (t_i - t_{i-1}) \sigma_{t_i-1,t_i}^2.$$

Hence, if we are given the mean reversion rate k, and the conditional volatilities, $\sigma_{i-1,i}$, we can compute the unconditional volatilities, σ_{t_i}.

However, the linear regression (4) is valid for any lognormal variables. We do not need to assume a, k are constant or that $\sigma_{t-1,t} = \sigma$.

To obtain the probability in NR, assume a binomial density, $n_i = 1$ for all i, in HSS [eq(10)]. This gives

$$q_{i-1,r} = \frac{a_i + b_i x_{i-1,r} - (i-1-r)\ln(u_i) - (r+1)\ln(d_i)}{[\ln(u_i) - \ln(d_i)]}, \tag{5}$$

However,

$$\begin{aligned} x_t^- &= (i-1-r)\ln(u_i) + (r+1)\ln(d_i) \\ x_t^+ &= (i-r)\ln(u_i) + r\ln(d_i) \\ x_t^+ - x_t^- &= \ln(u_i) - \ln(d_i) \end{aligned}$$

Hence (5) is equivalent to (3) with $a_i + b_i x_{i-1,r} = x_{t-1} + \mu(x_{t-1})$.

In general, the probability of an up-move is given by HSS [eq(10)]

$$q_{i-1,r} = \frac{a_i + b_i x_{i-1,r} - (N_{i-1} - r)\ln(u_i) - (n_i + r)\ln(d_i)}{n_i[\ln(u_i) - \ln(d_i)]}, \tag{6}$$

where $N_i = \sum_1^i n_l$. An example: Let $n_i = 2$, for all i, then for $i = 2$ and $r = 0$, we have

$$q_{1,0} = \frac{a_2 + b_2 x_{1,0} + (2-0)\ln(u_2) - 2\ln(d_2)}{2[\ln(u_2) - \ln(d_2)]},$$

Proposition 5.1 *(HSS) Suppose that u_i and d_i are chosen by*

$$d_i = \frac{2}{1 + e^{2\sigma_{i-1,i}\sqrt{\frac{t_i - t_{i-1}}{n_i}}}}$$

$$u_i = 2 - d_i$$

and the probability of an up move is

$$q_{i-1,r} = \frac{a_i + b_i x_{i-1,r} - (N_{i-1} - r)\ln(u_i) - (n_i + r)\ln(d_i)}{n_i[\ln(u_i) - \ln(d_i)]},$$

where $N_i = \sum_1^i n_l$. Then $\hat{\mu} \to \mu$ and $\hat{\sigma} \to \sigma$, as $n \to \infty$.

Proof

See HSS (1995).

A Multivariate Extension of HSS

In Peterson and Stapleton (2002) the original two variable version of HSS (eq 13, p1140), is modified, extended (to three variables) and implemented. It is illustrated by pricing a 'Power Reverse Dual' a derivative that depends on the process for two interest rates and an exchange rate.

First, we assume, that

$$\begin{aligned} x_t &= \ln[X_t/E(X_t)], \\ y_t &= \ln[Y_t/E(Y_t)], \end{aligned}$$

follow mean reverting Ornstein-Uhlenbeck processes, where:

$$\begin{aligned} dx_t &= \kappa_1(\phi_t - x_t)dt + \sigma_x(t)dW_{1,t} \\ dy_t &= \kappa_2(\theta_t - y_t)dt + \sigma_y(t)dW_{2,t}, \end{aligned} \tag{7}$$

where $E(dW_{1,t}dW_{2,t}) = \rho dt$. In (7), ϕ_t and θ_t are constants and κ_1 and κ_2 are the rates of mean reversion of x_t and y_t respectively. As in Amin(1995), it is useful to re-write these correlated processes in the orthogonalized form:

$$\begin{aligned} dx_t &= \kappa_1(\phi_t - x_t)dt + \sigma_x(t)dW_{1,t} \\ dy_t &= \kappa_2(\theta_t - y_t)dt + \rho\sigma_y(t)dW_{1,t} + \sqrt{1-\rho^2}\sigma_y(t)dW_{3,t}, \end{aligned} \tag{8}$$

where $E(dW_{1,t}dW_{3,t}) = 0$. Then, rearranging and substituting for $dW_{1,t}$ in (43), we can write

$$dy_t = \kappa_2(\theta_t - y_t)dt - \beta_{x,y}\left[\kappa_1(\phi_t - x_t)\right]dt + \beta_{x,y}dx_t + \sqrt{1-\rho^2}\sigma_y(t)dW_{3,t}.$$

In this bivariate system, we treat x_t as an independent variable and y_t as the dependent variable. The discrete form of the system can be written as follows:

$$\begin{aligned} x_t &= \alpha_{x,t} + \beta_{x,t}x_{t-1} + \varepsilon_{x,t} \\ y_t &= \alpha_{y,t} + \beta_{y,t}y_{t-1} + \gamma_{y,t}x_{t-1} + \delta_{y,t}x_t + \varepsilon_{y,t}, \end{aligned} \tag{9}$$

Proposition 5.2 (Approximation of a Two-Variable Diffusion Process) *Suppose that X_t, Y_t follows a joint lognormal process, where $E_0(X_t) = 1, E_0(Y_t) = 1\ \forall t$, and where*

$$\begin{aligned} x_t &= \alpha_{x,t} + \beta_{x,t} x_{t-1} + \varepsilon_{x,t} \\ y_t &= \alpha_{y,t} + \beta_{y,t} y_{t-1} + \gamma_{y,t} x_{t-1} + \delta_{y,t} x_t + \varepsilon_{y,t} \end{aligned}$$

Let the conditional logarithmic standard deviation of J_t be denoted as $\sigma_j(t)$ for $J = (X, Y)$, where

$$\sigma_j^2(t) = \mathrm{var}(\varepsilon_{j,t}) \tag{10}$$

If J_t is approximated by a log-binomial distribution with binomial density $N_t = N_{t-1} + n_t$ and if the proportionate up and down movements, u_{j_t} and d_{j_t} are given by

$$\begin{aligned} d_{j_t} &= \frac{2}{1 + \exp(2\sigma_j(t)\sqrt{\tau_t/n_t})} \\ u_{j_t} &= 2 - d_{j_t} \end{aligned}$$

and the conditional probability of an up-move at node r of the lattice is given by

$$q_{j_{t-1},r} = \frac{E_{t-1}(j_t) - (N_{t-1} - r)\ln(u_{j_t}) - (n_t + r)\ln(d_{j_t})}{n_t[\ln(u_{j_t}) - \ln(d_{j_t})]}$$

then the unconditional mean and volatility of the approximated process approach their true values, i.e., $\hat{E}_0(J_t) \to 1$ and $\hat{\sigma}_{j,t} \to \sigma_{j,t}$ as $n \to \infty$.

Steps in HSS: Single Factor Tree ($n = 1$ case)

Assume we are given b in the regression (mean reversion):

$$x_i = a_i + bx_{i-1} + \varepsilon_i$$

Also, we are given the local volatilities $\sigma_{i-1,i}$.

1. Compute
$$d_i = \frac{2}{1 + e^{2\sigma_{i-1,i}\sqrt{t_i - t_{i-1}}}}$$
$$u_i = 2 - d_i$$

2. Compute the nodal values for the unit mean tree
$$u_i^{i-r} d_i^r$$

3. Compute the unconditional voltilities using
$$t_i \sigma_{0,i}^2 = b^2 t_{i-1} \sigma_{0,i-1}^2 + (t_i - t_{i-1}) \sigma_{i-1,i}^2$$
starting with $i = 1$.

4. Compute the constant coefficients:
$$a_i = -\frac{1}{2} t_i \sigma_{0,i}^2 + b \frac{1}{2} t_{i-1} \sigma_{0,i-1}^2$$

5. Compute the probabilities
$$q_{i-1,r} = \frac{a_i + bx_{i-1,r} - (i-1-r)\ln(u_i) - r\ln(d_i) - \ln(d_i)}{\ln(u_i) - \ln(d_i)},$$

6. Given the unconditional expectations $E_0(X_i)$ compute the nodal values
$$X_{i,r} = E_0(X_i) u_i^{i-r} d_i^r$$

6 Interest-rate Models

6.1 No-arbitrage and Equilibrium Models

Equilibrium Interest-rate Models

An equilibrium interest-rate model assumes a stochastic process for the interest rate and derives a process for bond prices, assuming a value for the market price of risk.

No-arbitrage Interest-rate Models

A no-arbitrage interest-rate model assumes the current term structure of bond prices and builds a process for interest rates (and bond prices) that is consistent with this given term structure. In a no-arbitrage model, no bond can stochasticaly dominate another.

Proposition 6.1 **[No-Arbitrage Condition]** *A sufficient condition for no arbitrage is that the forward price of a zero-coupon bond is given by*

$$E_t(B_{t+1,T}) = \frac{B_{t,T}}{B_{t,t+1}}$$

where the expectation is taken under the risk-neutral measure.

Examples:

1. The Vasicek (1977) model (Equilibrium Model)

   - Assumes short rate (r_t) follows a normal distribution process
   - Assumes that short rate mean reverts at a constant rate
   - Derives equilibrium bond prices for all maturities

   $$dr_t = \kappa(a - r_t)dt + \sigma' dz.$$

   In discrete form:

   $$r_t - r_{t-1} = k(a - r_{t-1}) + \sigma' \varepsilon_t,$$

2. The Ho-Lee model

   - Assumes that the zero-coupon bonds follow a log-binomial process.
   - This implies that the short rate (r_t) follows a normal distribution process, in the limit.
   - Takes bond prices, and hence forward prices, (at $t = 0$) as given.
   - The model builds a process for the forward prices of the set of zero-coupon bonds.
   - No-arbitrage model, prices European-style bond options

3. The Black-Karasinski model

   - Assumes short rate (r_t) follows a lognormal distribution process
   - It derives from a prior model, the Black-Derman-Toy model, which did not have mean reversion.

$$d\ln(r_t) = \kappa[\theta(t) - \ln(r_t)]dt + \sigma(t)dz.$$

$$\ln(r_t) - \ln(r_{t-1}) = k[\theta(t) - \ln(r_t)] + \varepsilon_t$$

   - Takes bond prices, or futures rates (at $t = 0$) as given
   - No-arbitrage model, prices European-style, American-style bond options
   - Unconditional volatility (caplet vol) in the BK model:

$$var[\ln(r_t)] = (1-k)^2 var[\ln(r_{t-1})] + var(\varepsilon_t)$$

$$\sqrt{t}\sigma_{0,t} = (1-k)\sqrt{t-1}\sigma_{0,t-1} + \sigma_{t-1,t}$$

A Recombining BK model using HSS

To use the HSS method we follow the steps:

1. Given the local volatilities, $\sigma(t)$, and the mean reversion, k, we first build a tree of x_t, with $E_0(x_t) = 1$, for all x_t.

2. Then multiply by the expectations of r_t under the risk-neutral measure. The following result establishes that these expectations are the futures *LIBOR*, $h_{0,t}$

The following lemma states that, given the definition of the *LIBOR* futures contract, the futures *LIBOR* is the expected value of the spot rate, under the risk-neutral measure.

Lemma 8 (Futures *LIBOR*) *In a no-arbitrage economy, the time-t futures LIBOR, for delivery at T, is the expected value, under the risk-neutral measure, of the time-T spot LIBOR, i.e.*

$$f_{t,T} = E_t(r_T)$$

Also, if r_T is lognormally distributed under the risk-neutral measure, then:

$$\ln(f_{t,T}) = E_t[\ln(r_T)] + \frac{\mathrm{var}_t[\ln(r_T)]}{2},$$

*where the operator "*var*" refers to the variance under the risk-neutral measure.*

Proof

The price of the futures *LIBOR* contract is by definition

$$F_{t,T} = 1 - f_{t,T} \tag{11}$$

and its price at maturity is

$$F_{T,T} = 1 - f_{T,T} = 1 - r_T. \tag{12}$$

From Cox, Ingersoll and Ross (1981), the futures price $F_{t,T}$ is the value, at time t, of an asset that pays

$$V_T = \frac{1 - r_T}{B_{t,t+1}B_{t+1,t+2}...B_{T-1,T}} \tag{13}$$

at time T, where the time period from t to $t+1$ is one day. In a no-arbitrage economy, there exists a risk-neutral measure, under which the time-t value of the payoff is

$$F_{t,T} = E_t(V_T B_{t,t+1} B_{t+1,t+2} ... B_{T-1,T}). \tag{14}$$

Substituting (13) in (14), and simplifying then yields

$$F_{t,T} = E_t(1 - r_T) = 1 - E_t(r_T). \tag{15}$$

Combining (15) with (11) yields the first statement in the lemma. The second statement in the lemma follows from the assumption of the lognormal process for r_T and the moment generating function of the normal distribution. □

Lemma 8 allows us to substitute the futures rate directly for the expected value of the *LIBOR* in the process assumed for the spot rate. In particular, the futures rate has a zero drift, under the risk-neutral measure.

The Vasicek Model

Proposition 6.2 [Mean and Variance in the Vasicek Model] *Assume that the short-term interest rate is given by*

$$d\, r_t = \kappa(a - r_t) + \sigma\, dz$$

where dz is normally distributed with zero mean and unit variance. Then the conditional mean of r_s is

$$E_t(r_s) = a + (r_t - a)e^{-\kappa(s-t)}, \;\; t \leq s$$

and the conditional variance of r_s is

$$var_t(r_s) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(s-t)}), \;\; t \leq s$$

A Classification of Spot-Rate Models

Assume that the short-term rate of interest follows the GCEV process

$$dr_t = \mu(r_t,t)r_t dt + \sigma(t)r_t^{\gamma} dz.$$

1. If $\gamma = 0$, $\mu(r_t,t)r_t = \kappa(a - r_t)$, $\sigma(t) = \sigma$,

   $$dr_t = \kappa(a - r_t) + \sigma dz$$

   as in Vasicek (true process) and Hull-White (risk-neutral process).

   Extensions: Hull-White two-factor model.

2. If $\gamma = 1$, $\mu(r_t,t) = \mu$,

   $$dr_t = \mu r_t dt + \sigma(t) dz$$

   as in Black, Derman and Toy model (risk-neutral process)

   $$dln(r_t) = \kappa[\theta(t) - \ln(r_t)]dt + \sigma(t)dz$$

   as in Black-Karasinski model.

   Extensions: Peterson, Stapleton, Subrahmanyam two-factor model.

3. If $\gamma = 0.5$, $\mu(r_t,t)r_t = \alpha(\theta - r_t)$,

   $$dr_t = \alpha(\theta - r_t) + \sigma\sqrt{r_t}$$

   as in CIR model.

   Extensions: Credit risk factor, stochastic volatility models.

The PSS Two-Factor model

Hull and White (JD, 1994) suggest a class of two-factor models, where a function $f(r)$ follows a process with a stochastic conditional mean. PSS develop the special case where $f(r) = \ln(r)$. This gives a two-factor extension of the BK model. They define r_t as $LIBOR$ at time T: where

$$B_{t,t+m} = \frac{1}{1 + r_t m}$$

Solving the model they show that

$$\ln(r_t) - \ln(f_{0,t}) = \alpha_{r_t} + [\ln(r_{t-1}) - \ln(f_{0,t-1})](1-b) + \ln(\pi_{t-1}) + \varepsilon_t$$

where

$$\ln(\pi_t) = \alpha_{\pi_t} + \ln(\pi_{t-1})(1-c) + \nu_t,$$

under the risk-neutral measure.

To implement the model, PSS form the equations:

$$\begin{aligned} x_t &= \alpha_{x,t} + \beta_{x,t}x_{t-1} + y_{t+1} + \varepsilon_{x,t} \\ y_t &= \alpha_{y,t} + \beta_{y,t}y_{t-1} + \gamma_{y,t}x_{t-1} + \delta_{y,t}x_t + \varepsilon_{y,t} \end{aligned}$$

where $x_t = \ln\left(\frac{r_t}{f_{0,t}}\right)$.

Using HSS (NR), PSS choose

$$q_{x_{t-1},r} = \frac{E_{t-1}(x_t) - (N_{t-1} - r)\ln(u_{x_t}) - (n_t + r)\ln(d_{x_t})}{n_t[\ln(u_{x_t}) - \ln(d_{x_t})]}$$

where

$$E_{t-1}(x_t) = \alpha_{x,t} + \beta_{x,t}x_{t-1} + y_{t+1}$$

In this model, the no-arbitrage condition [futures = expected spot] is gauranteed by choosing the appropriate q on the tree of rates. The model is then used to price Bermudan-style swaptions and yield-spread options.

7 The Ho-Lee Model

Features of the model

- The model prices interest-rate derivatives, given the current term-structure of bond prices, and given a binomial process for the term-structure evolution
- One-factor (any bond or interest rate) generates the whole term structure
- It is analogous to the Cox, Ross, Rubinstein (limit Black-Scholes) model for bond options
- The model is Arbitrage-Free (AR)

<u>Notation</u>

$B_{t,T,i} = B_{t,i}(T)$ is the discount function in state i at time t, where i is the number of up-moves of the process. The discount function follows a two-state (binomial) process. p is the risk-neutral probability of an up move.

$$\begin{tikzcd}
 & B_{t+1,i+1}(.) \\
B_{t,i}(.) \arrow[ur, no head] \arrow[dr, no head] & \\
 & B_{t+1,i}(.)
\end{tikzcd}$$

Let $u(T)$ and $d(T)$ be T-dimensional 'perturbation functions' defined by

$$\begin{aligned}
B_{t+1,i+1}(T) &= \frac{B_{t,i}(T+1)}{B_{t,i}(1)}u(T) \\
B_{t+1,i}(T) &= \frac{B_{t,i}(T+1)}{B_{t,i}(1)}d(T)
\end{aligned}$$

Proposition 7.1 [Ho-Lee Process]

1. *A constant, time-independent risk-neutral probability p exists and for any T*
$$p = \frac{1 - d(T)}{u(T) - d(T)}$$
2. *The process recombines only if a δ exists such that*
$$u(T) = \frac{1}{p + (1-p)\delta^T}$$

Proof

a) Form a portfolio with 1 bond of maturity T and α bonds of maturity τ. The cost of the portfolio is, at time t, is $B_T + \alpha B_\tau$ (dropping subscripts t, i). The return on the portfolio in the up state at $t+1$ is
$$\frac{B_T}{B_1} u(T-1) + \alpha \frac{B_\tau}{B_1} u(\tau - 1)$$
In the down-state it is
$$\frac{B_T}{B_1} d(T-1) + \alpha \frac{B_\tau}{B_1} d(\tau - 1)$$
Choose $\alpha = \alpha^*$ so that these are equal, that is
$$\alpha^* = \frac{d(T-1)B_T - u(T-1)B_T}{u(\tau-1)B_\tau - d(\tau-1)B_\tau} = \frac{B_T[d(T-1) - u(T-1)]}{B_\tau[u(\tau-1) - d(\tau-1)]}$$
With $\alpha = \alpha^*$, the discounted value of the return must equal the cost, hence
$$B_T + \alpha^* B_\tau = B_T[d(T-1)] + [\alpha^* d(\tau-1)]B_\tau$$
and this implies
$$\frac{1 - d(T)}{u(T) - d(T)} = p$$
which is a constant (for a proof, see exercise 8.1)

b)

$$
\begin{tikzcd}
 & B_{t+1,i+1}(T+1) \\
B_{t,i}(T+2) \arrow[ur, dash] \arrow[dr, dash] & \\
 & B_{t+1,i}(T+1)
\end{tikzcd}
$$

Recombination means that

$$
\begin{aligned}
B_{t+2,i+1}(T) &= \frac{B_{t+1,i+1}(T+1)}{B_{t+1,i+1}(1)} d(T) = \frac{B_{t+1,i}(T+1)}{B_{t+1,i}(1)} u(T) \\
B_{t+2,i+1}(T) &= \frac{\frac{B_{t,i}(T+2)}{B_{t,i}(1)} u(T+1)}{\frac{B_{t,i}(2)}{B_{t,i}(1)} u(1)} d(T) = \frac{\frac{B_{t,i}(T+2)}{B_{t,i}(1)} d(T+1)}{\frac{B_{t,i}(2)}{B_{t,i}(1)} d(1)} u(T)
\end{aligned}
$$

It follows that

$$u(T+1)d(T)d(1) = d(T+1)u(T)u(1),$$

for all T. Hence,

$$u(T-1)\left[\frac{1-pu(T)}{1-p}\right]\left[\frac{1-pu(1)}{1-p}\right] = \left[\frac{1-pu(T+1)}{1-p}\right]u(T)u(1)$$

and simplifying yields

$$\frac{1}{u(T+1)} = \frac{\delta}{u(T)} + \gamma$$

where

$$\gamma = \frac{p\left[u(1)-1\right]}{(1-p)u(1)}.$$

The solution to this difference equation, with $u(0) = 1$ is

$$u(T) = \frac{1}{p + (1-p)\delta^T}$$

and using part a),

$$d(T) = \frac{\delta^T}{p + (1-p)\delta^T}.$$

Proposition 7.2 [Contingent Claims in the Ho-Lee Model] *Consider a contingent claim paying $C(t,i)$ at time t, in state i, then its value at time $t-1$ is*

$$C(t-1,i) = \{p[C(t,i+1)] + (1-p)[C(t,i)]\}B_{t-1,i}$$

Proof

Form a portfolio of one discount bond with maturity t plus α contingent claims. Choose α so that the portfolio is risk free. The result then follows as in CRR (1979).

Note, if we know the process for $B_t(1)$ and p, we can price any contingent claim. This is a one-factor model result.

Steps for Constructing the Ho-Lee Model

1. Use market data to estimate the set of zero-coupon bond prices at $t = 0$.

2. Use forward parity to compute the one-period-ahead forward prices at $t = 0$, for each bond, $B_{0,1,n}$, where

$$B_{0,1,n} = \frac{B_{0,n}}{B_{0,1}}$$

3. Compute the up and down movements $u(T)$ and $d(T)$ for times to maturity $T = 1, 2, ..., n$, where

$$d(T) = \frac{\delta^T}{0.5(1 + \delta^T)}$$

$$u(T) = 2 - d(T)$$

4. Compute $B^u_{1,n}$ in the up-state using

$$B^u_{1,n} = B_{0,1,n} u(n-1)$$

   Then compute $B^d_{1,n}$ in the down-state using

$$B^d_{1,n} = B_{0,1,n} d(n-1)$$

5. Compute the set of forward prices at $t = 1$ in the up-state, $B^u_{1,2,n}$, using forward parity. Then compute the set of forward prices at $t = 1$ in the down-state, $B^d_{1,2,n}$.

6. Starting in the up-state at $t = 1$ compute $B^{uu}_{2,n}$ (in the up-up state at $t = 2$) using the method in step 4, then compute $B^{ud}_{2,n}$ and $B^{dd}_{2,n}$.

7. After step 6 you should have a term structure of zero-coupon bond prices at each date and in each state. Use these to compute interest rates (yields for example) or coupon bond prices, as required:

(a) Use

$$B^s_{t,n} = \frac{1}{(1 + y^s_{t,n})^{n-t}}$$

to compute the $n - t$ year maturity yield rate in state s at time t.

(b) Use

$$B^{c,s}_{t,n} = cB^s_{t,1} + cB^s_{t,2} + ...cB^s_{t,n} + B^s_{t,n}$$

to compute the price of an $n - m$ maturity bond, with coupon c, in state s.

8. Compute the price of an interest-rate derivative by starting at the maturity date of the derivative, working out the expected value using the probability $p = 0.5$, and discounting by the one-period zero-coupon bond price, using

$$C^s_t = \left[C^{s+1}_{t+1} 0.5 + C^s_{t+1} 0.5\right] B^s_{t,1}$$

where s indicates the state at time t by the number of up-moves of the process from 0 to t.

8 The *LIBOR* Market Model

8.1 Origins of the LMM

- Forward Rate Models (HJM) and Forward Price Models (Ho-Lee)
- Black Model for Caplet Pricing

Brace, Gatarek and Musiela (BGM) and Miltersen, Sandmann and Sonderman (MSS) build a forward *LIBOR* model consistent with the Black Model holding for each Caplet. Note that the BK model is not consistent with the Black model (in spite of its lognormal assumption).

Heath-Jarrow-Morton, Forward-Rate Models

HJM models build the process for the forward interest rate. Similar to Ho-Lee, but forward rate, not forward price. For example, the Brace-Gatarak-Musiela (BGM) model builds a process for the forward *LIBOR*. Usually assume a convenient volatility process (ex. constant vol). The models are used for pricing complex interest-rate derivatives.

Proposition 8.1 **[The Black Model: Interest-Rate Caplet]**

$$caplet_t = \frac{A}{1 + f_{t,t+T}\delta}\delta[f_{t,t+T}N(d_1) - kN(d_2)]B_{t,t+T}$$

where

$$d_1 = \frac{ln(\frac{f_{t,t+T}}{k}) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Main Features of the LMM

The main features of the LMM are as follows:

- Forward rates are *conditional* lognormal over each discrete period of time.
- The first input is the term structure of forward rates at time $t = 0$.
- This complete term structure of forward rates is perturbed over each time period, t
- The methodology is similar to Ho-Lee, but uses forward rates rather than forward bond prices
- The interest rate generated is usually 3-month *LIBOR*.

8.2 No-Arbitrage Pricing

We start by considering some implications of no-arbitrage. First, we assume the following no-arbitrage relationships hold, where expectations are taken under the risk-neutral measure. We also assume that the zero-coupon bond prices $B_{t,t+1}$ are stochastic. For convenience, write E_0 as E.

Lemma 9 (No-Arbitrage Pricing) *If no dividend is payable on an asset:*

1. *the spot price of the asset is*

$$S_0 = B_{0,1}E[B_{1,2}E_1[B_{2,3}E_2[...B_{t-1,t}E_{t-1}(S_t)]]]$$

2. *and the t-period forward price of the asset*

$$F_{0,t} = S_0/B_{0,t}$$

Proposition 8.2 (Zero-Coupon Bond Forward Prices) *When expectations are taken under the risk-neutral measure: The t-period forward price of a $t+T$-period maturity zero-coupon bond is*

$$E(B_{1,t,t+T}) - B_{0,t,t+T} = -\frac{B_{0,1}}{B_{0,t}} cov(B_{1,t,t+T}, B_{1,t})$$

<u>*Proof*</u>

From no-arbitrage:

$$\begin{aligned} B_{0,t} &= B_{0,1}E(B_{1,t}), \\ B_{0,t+T} &= B_{0,1}E(B_{1,t+T}). \end{aligned}$$

From forward parity:

$$B_{1,t+T} = B_{1,t,t+T}B_{1,t},$$

hence taking expectations and using the definition of covariance,

$$E(B_{1,t+T}) = E(B_{1,t,t+T})E(B_{1,t}) + cov(B_{1,t,t+T}, B_{1,t})$$

Substituting for the expected bond prices

$$\frac{B_{0,t+T}}{B_{0,1}} = E(B_{1,t,t+T})\frac{B_{0,t}}{B_{0,1}} + cov(B_{1,t,t+T}, B_{1,t})$$

and multiplying by $B_{0,1}$ and deviding by $B_{0,t}$ yields

$$\frac{B_{0,t+T}}{B_{0,t}} = E(B_{1,t,t+T}) + \frac{B_{0,1}}{B_{0,t}} cov(B_{1,t,t+T}, B_{1,t})$$

and since, from forward parity

$$\frac{B_{0,t+T}}{B_{0,t}} = B_{0,t,t+T}$$

then we have

$$E(B_{1,t,t+T}) - B_{0,t,t+T} = -\frac{B_{0,1}}{B_{0,t}} cov(B_{1,t,t+T}, B_{1,t})$$

Corollary 5 *One-Period Ahead Forward Prices*

Let $t = 1$*, then*

$$E(B_{0,T+1}) - B_{0,1,T+1}$$

and hence

$$B_{0,1,T+1} = E(B_{1,2}B_{1,2,T+1})$$

Also, with $T = 1$*,*

$$B_{0,1,2} = E(B_{1,2})$$

8.3 The *LIBOR* Market Model: Notation

- $B_{t,t+\delta}$ = Value at t of a zero-coupon bond paying 1 unit of currency at $t+\delta$.
- δ = Interest-rate reset interval (ex. 3 months) as a proportion of a year
- $B_{t,t+T}$ = Value at t of a zero-coupon bond paying 1 unit of currency at $t+T$.
- $B_{t,t+T,t+T+\delta}$ = Forward price at t for delivery of a zero-coupon bond (with maturity δ) at T.
- $f_{t,t+T}$ = T-period forward *LIBOR* at time t if $T = 0$, $f_{t,t}$ is the spot *LIBOR* at t.
- Note that in this notation

$$B_{t,t+T,t+T+\delta} = \frac{1}{1 + f_{t,t+T}\delta}$$

Definition 8.1 *A Forward Rate Agreement (FRA) on δ-periodLIBOR, with maturity t, has a payoff*

$$\frac{(f_{t,t}-k)\delta}{1+f_{t,t}\delta}$$

at date t.

Proposition 8.3 *(Drift of the One-Period Forward rate)*

Since a one-period FRA struck at the forward rate $f_{0,1}$ has a zero value:

$$E\left[\frac{(f_{1,1}-f_{0,1})\delta}{1+\delta f_{1,1}}\right]=0.$$

It follows that

$$E\left(\frac{\delta f_{1,1}}{1+\delta f_{1,1}}\right)=\frac{\delta f_{0,1}}{1+\delta f_{0,1}}.$$

Also

$$E(\delta f_{1,1})-\delta f_{0,1}=-cov\left(\delta f_{1,1},\frac{1}{1+\delta f_{1,1}}\right)(1+\delta f_{0,1})\geq 0$$

Hence, the drift of the forward rate is given by

$$E(f_{1,1})-f_{0,1}=-\left(\frac{1}{\delta}\right)cov\left(\delta f_{1,1},\frac{1}{1+\delta f_{1,1}}\right)(1+\delta f_{0,1})\geq 0$$

Proof

Expanding the lhs of the second equation, using the definition of covariance and employing Corollary 5 yields the Proposition.

Proposition 8.4 (Drift of Two-Period Forward) *Since a two-period FRA has a zero value:*

$$E\left[\left(\frac{\delta(f_{1,2}-f_{0,2})}{1+\delta f_{2,2}}\right)\frac{1}{1+\delta f_{1,1}}\right]=0.$$

It follows that

$$E(f_{1,2})-f_{0,2}=-\left(\frac{1}{\delta}\right)cov\left[\delta f_{1,2},\frac{1}{1+\delta f_{1,1}}\cdot\frac{1}{1+\delta f_{1,2}}\right](1+\delta f_{0,1})(1+\delta f_{0,2})$$

Proof

Expanding the lhs, using the definition of covariance and employing Corollary 5 yields the Proposition.

Lemma 10 (Covariances and Covariances of Logarithms) *From Taylor's Theorem we can write*

$$\ln X=\ln a+\frac{1}{a}(X-a)+...$$
$$\ln Y=\ln b+\frac{1}{b}(Y-b)+...$$

Hence

$$cov(\ln X,\ln Y)\approx\frac{1}{a}\frac{1}{b}cov(X,Y)$$

Applying this we have for example:

$$cov(\ln(f_{1,t}),\ln(f_{1,\tau}))\approx\frac{1}{f_{0,t}}\frac{1}{f_{0,\tau}}cov(f_{1,t},f_{1,\tau})$$

Lemma 11 (Stein's Lemma) *For joint normal variables, x,y:*

$$cov(x,g(y))=E(g'(y))cov(x,y)$$

Hence, if $x=\ln X$ and $y=\ln Y$ Then

$$cov\left(\ln X,\ln\left(\frac{1}{1+Y}\right)\right)=E\left[\frac{-Y}{1+Y}\right]cov\left(\ln X,\ln Y\right)$$

Proposition 8.5 *(Drift of the One-Period Forward rate)*

From Proposition 8.3 we have

$$E(f_{1,1}) - f_{0,1} = -\left(\frac{1}{\delta}\right) cov\left(\delta f_{1,1}, \frac{1}{1+\delta f_{1,1}}\right)(1+\delta f_{0,1})$$

Using Lemma 10 and Lemma 11 we have

$$E(f_{1,1}) - f_{0,1} = cov\left[\ln(f_{1,1}), \ln(f_{1,1})\right] \frac{f_{0,1}\delta f_{0,1}}{1+f_{0,1}\delta}$$

and the annualised drift of the one-period forward rate is

$$E(f_{1,1}) - f_{0,1} = \delta\sigma_{0,0} f_{0,1} \frac{\delta f_{0,1}}{1+\delta f_{0,1}}$$

Proof

For notational simplicity, we write $f_{t,t+T}\delta$ as $f_{t,t+T}$. First, consider the drift of the one-period forward. From Proposition 8.3 we have

$$E_0(f_{1,1}) - f_{0,1} = -cov\left(f_{1,1}, \frac{1}{1+f_{1,1}}\right)(1+f_{0,1})$$

Using lemma 10

$$cov\left(f_{1,1}, \frac{1}{1+f_{1,1}}\right) = cov\left[\ln(f_{1,1})\ln\left(\frac{1}{1+f_{1,1}}\right)\right] f_{0,1}/(1+f_{0,1}),$$

Hence,

$$E_0(f_{1,1}) - f_{0,1} = -cov\left[\ln(f_{1,1})\ln\left(\frac{1}{1+f_{1,1}}\right)\right]/f_{0,1}).$$

Now using Lemma 11

$$cov\left(\ln(f_{1,1}), \ln\left(\frac{1}{1+f_{1,1}}\right)\right) = cov\left[\ln(f_{1,1}), \ln(f_{1,1})\right]\left[\frac{-f_{0,1}}{1+f_{0,1}}\right],$$

and hence

$$E(f_{1,1}) - f_{0,1} = cov\left[\ln(f_{1,1}), \ln(f_{1,1})\right] \frac{f_{0,1}f_{0,1}}{1+f_{0,1}}.$$

Finally, remembering that $f_{t,t+T}\delta$ was written as $f_{t,t+T}$,

$$E(\delta f_{1,1}) - \delta f_{0,1} = cov\left[\ln(\delta f_{1,1}), \ln(\delta f_{1,1})\right]\frac{\delta f_{0,1}\delta f_{0,1}}{1+\delta f_{0,1}}.$$

and

$$E(f_{1,1}) - f_{0,1} = cov\left[\ln(f_{1,1}), \ln(f_{1,1})\right]\frac{f_{0,1}\delta f_{0,1}}{1+f_{0,1}\delta}.$$

Now if we define the volatility of the forward rate on an annualised basis, by

$$\delta\sigma_T^2 = var_t[\ln(f_{t,t+T})]$$

the annualised drift of the forward rate is, where δ is the length of the time step,

$$\frac{E(f_{1,1}) - f_{0,1}}{f_{0,1}} = \delta\sigma_{0,0}\frac{\delta f_{0,1}}{1+\delta f_{0,1}}$$

Proposition 8.6 *(Drift of the Two-Period Forward rate)*

Consider the drift of the two-period forward rate, from Proposition 8.4

$$E(f_{1,2}) - f_{0,2} = -\left(\frac{1}{\delta}\right) cov \left[\delta f_{1,2}, \frac{1}{1+\delta f_{1,1}} \cdot \frac{1}{1+\delta f_{1,2}}\right] (1+\delta f_{0,1})(1+\delta f_{0,2})$$

Using Lemma 10 and Lemma 11 we have

$$\frac{E(f_{1,2}) - f_{0,2}}{f_{0,2}} = \delta \left[\sigma_{0,1} \frac{\delta f_{0,1}}{1+\delta f_{0,1}} + \sigma_{1,1} \frac{\delta f_{0,1}}{1+\delta f_{0,1}}\right]$$

Proposition 8.7 *The BGM Model*

$$\frac{E(f_{1,T}) - f_{0,T}}{f_{0,T}} = \delta \left[\frac{\delta f_{0,1}}{1+\delta f_{0,1}} \sigma_{0,T-1} + \frac{\delta f_{0,2}}{1+\delta f_{0,2}} \sigma_{1,T-1} + \cdots + \frac{\delta f_{0,T}}{1+\delta f_{0,T}} \sigma_{T-1,T-1}\right]$$

and given time homogeneous covariances:

$$\frac{E(f_{t+1,t+T}) - f_{t,t+T}}{f_{t,t+T}} = \delta \left[\frac{\delta f_{t,t+1}}{1+\delta f_{t,t+1}} \sigma_{0,T-1} + \frac{\delta f_{t,t+2}}{1+\delta f_{t,t+2}} \sigma_{1,T-1} + \cdots + \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}} \sigma_{T-1,T-1}\right]$$

9 Implementing and Calibrating the LMM

9.1 The Yield Curve

As in the Ho-Lee Model (and all HJM models), the model inputs the initial term structure of zero-coupon bond prices, or forward *LIBOR*. We assume that the forward *LIBOR* curve is available with maturities equal to each re-set date Note that this is in contrast with the BK model, which requires iteration to match the yield curve (or inputs the futures rates).

Given the notation $f_{t,t+T}$ the initial forward curve input is

$$f_{0,T}, \quad T = 0, 1, 2, ...N - 1$$

where the reset intervals are indexed $1, 2, ..., N - 1$

9.2 Caplet Volatilities and Forward Volatilities

Definitions

We have to be careful since there are several different definitions of volatility. These come from:

1. Variance of bond prices

$$\mathrm{var}_{t-1}(\ln B_{t,t+T})$$

   This is bond price volatility (used by BGM and Hull, ch 24).

2. conditional variance of *LIBOR*

$$\mathrm{var}_{t-1}(\ln r_t)$$

   This is local volatility (as in the BK model)

3. Unconditional variance of *LIBOR*

$$\mathrm{var}_0(\ln r_t)$$

   This is the unconditional volatility of *LIBOR* often referred to as the 'caplet volatility' since it can be estimated from cap prices.

4. Variance of forward *LIBOR*

$$\mathrm{var}_{t-1}(\ln f_{t,t+T})$$

This is the (local) volatility of the forward *LIBOR* rate

Notation

- Caplet Volatilities

  $$\mathrm{capvol}_{t,T}$$

  is the caplet volatility (annualised) observed at t for caplets with maturity $t+T$.

- Forward *LIBOR* volatilities

  $$\mathrm{fvol}_{t,T}$$

  is the volatility (annualised) of the T th forward rate, at time t

  However, we can drop the subscript t, if we assume that forward vols depend only on the maturity of the forward, as in Ho-Lee. Then we denote the volatility as σ_T.

- In the multi-factor LMM, we will use

  $$\sigma_T(i)$$

  for the volatility at time t of the T th forward arising from the i th factor.

The Relationship Between Caplet Vols and Forward Vols

The forward rates follow an approximate random walk. Hence,

$$\begin{aligned}
T\mathrm{capvol}_T^2 &= \mathrm{fvol}_{0,T-1}^2 + \mathrm{fvol}_{1,T-2}^2 + ... + \mathrm{fvol}_{T-1,0}^2 \\
(T-1)\mathrm{capvol}_{T-1}^2 &= \mathrm{fvol}_{0,T-2}^2 + \mathrm{fvol}_{1,T-3}^2 + ... + \mathrm{fvol}_{T-2,0}^2 \\
... &= ... \\
1\mathrm{capvol}_1^2 &= \mathrm{fvol}_{0,0}^2
\end{aligned}$$

Computing Forward Volatilities

The equations above can be solved for the forward vols only if additional restrictions are imposed. A reasonable assumption, may be to assume time homogenous forward volatilities, as in the Ho-Lee model. If we assume that the volatilities are only dependent on the forward maturity T, and not on where we are in the tree, we have

$$\mathrm{fvol}_{1,T} = \mathrm{fvol}_{2,T} = ... = \mathrm{fvol}_{t,T} = \sigma_T$$

We can then solve the system of equations for the forward volatilities using the 'bootstrap' equations:

$$\begin{array}{rcl}
\mathrm{capvol}_1^2 & = & \sigma_0^2 \\
2\mathrm{capvol}_2^2 & = & \sigma_0^2 + \sigma_1^2 \\
3\mathrm{capvol}_3^2 & = & \sigma_0^2 + \sigma_1^2 + \sigma_2^2 \\
... & = & ... \\
T\mathrm{capvol}_T^2 & = & \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + ... + \sigma_{T-1}^2
\end{array}$$

9.3 The Factor Model and Forward Covariances

Assume that each forward rate is generated by a factor model with I independent factors:

$$f_{t,t+T} = f_{t-1,t+T} + d_{t-1,t+T} + \sum_{i=1}^{I} \lambda_t(i)\sigma_T(i) f_{t-1,t+T}$$

where d is the drift per period. with the restriction:

$$\sum_{i=1}^{I} \sigma_T(i)^2 = \sigma_T^2$$

For example, if $I = 1$,

$$f_{t,t+T} = f_{t-1,t+T} + d_{t-1,t+T} + \lambda_t(1)\sigma_T f_{t-1,t+T}$$

If $I = 2$,

$$f_{t,t+T} = f_{t-1,t+T} + d_{t-1,t+T} + \lambda_t(1)\sigma_T(1)f_{t-1,t+T} + \lambda_t(2)\sigma_T(2)f_{t-1,t+T}$$

with the restriction:

$$\sigma_T(1)^2 + \sigma_T(2)^2 = \sigma_T^2$$

In this case

$$\begin{aligned} \frac{f_{t,t+T} - f_{t-1,t+T}}{f_{t-1,t+T}} &= d_{t-1,t+T}/f_{t-1,t+T} + \lambda_1\sigma_T(1) + \lambda_2\sigma_T(2) \\ \frac{f_{t,t+\tau} - f_{t-1,t+\tau}}{f_{t-1,t+\tau}} &= d_{t-1,t+\tau}/f_{t-1,t+\tau} + \lambda_1\sigma_\tau(1) + \lambda_2\sigma_\tau(2) \end{aligned}$$

It follows that

$$cov[\ln(f_{t,t+T}), \ln(f_{t,t+\tau})] = \delta\sigma_T(1)\sigma_\tau(1) + \delta\sigma_T(2)\sigma_\tau(2).$$

This equation allows us to compute the covariance matrix of the forward rates.

9.4 Steps for Building A One-Factor, Three-period LMM

Inputs

1. Input time-0 structure of forward *LIBOR* rates

$$f_{0,0}, f_{0,1}, f_{0,2}, f_{0,3}$$

2. Input time-0 structure of caplet volatilities

$$\text{capvol}_1,\ \text{capvol}_2, \text{capvol}_3$$

Computing Forward Volatilities

The forward volatilities solve the following 'bootstrap' equations:

$$\begin{aligned} \text{capvol}_1^2 &= \sigma_0^2 \\ 2\text{capvol}_2^2 &= \sigma_0^2 + \sigma_1^2 \\ 3\text{capvol}_3^2 &= \sigma_0^2 + \sigma_1^2 + \sigma_2^2 \end{aligned}$$

Computing Covariances

Compute array of $\sigma_{\tau,T}$, for $\tau = 1, 2, 3$ and $T = 1, 2, 3$, using

$$\sigma_{\tau,T} = \sigma_\tau \sigma_T \tag{16}$$

Building the Factor Binomial Trees

The binomial tree for the factor has an unconditional mean of 0 and a conditional variance of δ. Hence

$$\lambda_{t+1} = \pm\sqrt{\delta}$$

We have, assuming probabilities, $p = 0.5$,

$$E_t(\lambda_{t+1}) = 0$$

$$\mathrm{var}_t(\lambda_{t+1}) = \delta.$$

The Evolution of the Forward rates

Let $d_{t,t+T}$ denote the drift of the T th forward rate, $f_{t,t+T}$, from time t to time $t+1$. At $t=0$ we have:

$$\begin{aligned}
d_{0,0} &= \delta f_{0,1}\frac{\delta f_{0,1}}{1+\delta f_{0,1}}\sigma_{0,0} \\
d_{0,1} &= \delta f_{0,2}\left[\frac{\delta f_{0,1}}{1+\delta f_{0,1}}\sigma_{0,1} + \frac{\delta f_{0,2}}{1+\delta f_{0,2}}\sigma_{1,1}\right] \\
d_{0,2} &= \delta f_{0,3}\left[\frac{\delta f_{0,1}}{1+\delta f_{0,1}}\sigma_{0,2} + \frac{\delta f_{0,2}}{1+\delta f_{0,2}}\sigma_{1,2} + \frac{\delta f_{0,3}}{1+\delta f_{0,3}}\sigma_{2,2}\right]
\end{aligned}$$

and

$$\begin{aligned}
f_{1,1} &= f_{0,1} + d_{0,0} + \lambda_1\sigma_0 f_{0,1} \\
f_{1,2} &= f_{0,2} + d_{0,1} + \lambda_1\sigma_1 f_{0,2} \\
f_{1,3} &= f_{0,3} + d_{0,2} + \lambda_1\sigma_2 f_{0,3}
\end{aligned}$$

The drift from time 1 to time 2 is

$$\begin{aligned}
d_{1,1} &= \delta f_{1,2}\frac{\delta f_{1,2}}{1+\delta f_{1,2}}\sigma_{0,0} \\
d_{1,3} &= \delta f_{1,3}\left[\frac{\delta f_{1,2}}{1+\delta f_{1,2}}\sigma_{0,1} + \frac{\delta f_{1,3}}{1+\delta f_{1,3}}\sigma_{1,1}\right]
\end{aligned}$$

$$\begin{aligned}
f_{2,2} &= f_{1,2} + d_{1,1} + \lambda_2\sigma_0 f_{1,2} \\
f_{2,3} &= f_{1,3} + d_{1,2} + \lambda_2\sigma_1 f_{1,3}
\end{aligned}$$

The drift from time 2 to time 3 is

$$d_{2,2} = \delta f_{2,3}\frac{\delta f_{2,3}}{1+\delta f_{2,3}}\sigma_{0,0}$$

$$f_{3,3} = f_{2,3} + d_{2,2} + \lambda_3 \sigma_0 f_{2,3} \tag{17}$$

Bond Prices

First compute the spot one-period bond prices $B_{t,t+1,i}$. These are given by

$$B_{0,1} = \frac{1}{1 + \delta f_{0,0}}$$

$$B_{1,2} = \frac{1}{1 + \delta f_{1,1}},$$

$$B_{2,3} = \frac{1}{1 + \delta f_{2,2}},$$

$$B_{3,4} = \frac{1}{1 + \delta f_{3,3}},$$

Caplet Prices

The European-style Caplet is priced using the equations:

$$\begin{aligned}
C_3 &= \max(f_{3,3} - k, 0) A\delta B_{3,4} \\
C_2 &= E_2(C_3) B_{2,3} \\
C_1 &= E_1(C_2) B_{1,2} \\
C_0 &= E_0(C_1) B_{0,1}
\end{aligned}$$

A Bermudan-style Caplet is priced using:

$$\begin{aligned}
BM_3 &= \max(f_{3,3} - k, 0) A\delta B_{3,4} \\
BM_2 &= \max[(f_{2,2} - k) A\delta, E_2(BM_3) B_{2,3}] \\
BM_1 &= \max[(f_{1,1} - k) A\delta, E_1(BM_2) B_{2,3}] \\
BM_0 &= E_0(BM_1) B_{0,1}
\end{aligned}$$

Extending of the LMM to Two Factors

Hull shows how the model can be extended to two or more factors. Essentially, we allow the covariance matrix to be generated by two factors:

Computing Factor Loadings

1. Input constants $a_{1,0}, ...$ [for convenience, assume $a_{1,T} = (a_{1,0})^{T+1}$, then only input $a_{1,0}$.]

2. Compute the relative factor loadings for factor 2 using:
$$a_{2,T} = (1 - (a_{1,T})^2)^{0.5} \tag{18}$$

3. Compute the absolute factor loadings for factor 1 and 2 using:
$$\sigma_T(1) = a_{1,T}\sigma_T \tag{19}$$
$$\sigma_T(2) = a_{2,T}\sigma_T \tag{20}$$

Computing Covariances

Compute array of $\sigma_{\tau,T}$, for $\tau = 0, 1, ..., 20$ and $T = 0, 1, ..., 20$, using
$$\sigma_{\tau,T} = \sigma_\tau(1)\sigma_T(1) + \sigma_\tau(2)\sigma_T(2) \tag{21}$$

Building the Factor Binomial Trees

The binomial trees for factor 1, 2: $\lambda_{1,t}$ and $\lambda_{2,t}$ have an unconditional mean of 0 and a conditional variance of 1. Hence
$$\lambda_{1,t+1} = \pm\sqrt{\delta}$$
$$\lambda_{2,t+1} = \pm\sqrt{\delta}.$$

We have, assuming probabilities $p = 0.5$,
$$E(\lambda_{1,t+1}) = 0$$
$$\mathrm{var}_t(\lambda_{1,t+1}) = \delta.$$

$$f_{1,T} = f_{0,T} + d_{0,T} + \lambda_{1,1}\sigma_T(1)f_{0,T} + \lambda_{2,1}\sigma_T(2)f_{0,T} \tag{22}$$

9.5 The HSS Version of the LMM: A Re-combining Node Methodology

The models suggested in this section use the methodology suggested in Nelson and Ramaswamy, *RFS*, 1990, Ho, Stapleton and Subrahmanyam, *RFS*, 1995. The basic intuition: we first build a recombining binomial tree with the correct volatility characteristics. Then we adjust the probabilities of moving up the tree to reflect the correct drift of the process.

From Ito's lemma, the drift of $\ln x$ is :

$$d \ln x = \frac{dx}{x} - \frac{1}{2}\sigma^2$$

Hence, if dx is the drift in the process, we can compute the drift in the logarithm of the process.

For example, from $t = 0$ to $t = 1$, the drift in the zero th forward is

$$d_{0,0} = \delta f_{0,1} \frac{\delta f_{0,1}}{1 + \delta f_{0,1}} \sigma_{0,0}$$

and the drift of the logarithm is

$$m_{0,0} = d \ln(d_{0,0}) = \delta \left[\frac{f_{0,1}}{1 + f_{0,1}} \sigma_{0,0} - \frac{1}{2}\sigma_{0,0}^2 \right].$$

The probability, $q_{0,0}$, of an up-move (for the case of $n = 1$) has to satisfy:

$$q_{0,0} \ln(f_{1,1,u}) + (1 - q_{0,0}) \ln(f_{1,1,d}) = \ln(f_{0,1}) + m_{0,1}$$

Hence, if u_0 and d_0 are the proportionate up and down moves for a 0-period maturity forward rate, over the first period, we have

$$q_{0,0} = \frac{-\ln(d_0) + m_{0,0}}{\ln(u_0) - \ln(d_0)}$$

Now consider the drift from $t = 1$ to $t = 2$ of the forward $f_{1,2}$, assuming that we are at $f_{1,2,0}$.

The probability $p_{1,0}$ has to satisfy:

$$q_{1,0}\ln(f_{2,2,0}) + (1 - q_{1,0})\ln(f_{2,2,1}) = \ln(f_{1,2,0}) + m_{1,1,0}$$

Hence,

$$q_{1,0} = \frac{\ln(u_1) + m_{1,1,0} - \ln(u_0) - \ln(d_0)}{\ln(u_0) - \ln(d_0)}$$

9.6 Notes for Constructing the LMM-2-Factor Model Spreadsheet: HSS Method.

Forward Volatilities and Covariances

Inputs

1. Input time 0 structure of forward *LIBOR* rates

$$f_{0,T},\ T = 0, 1, ..., N$$

2. Input time 0 structure of caplet volatilities

$$\text{capvol}_t,\ \ t = 1, 2, .., N$$

Compute Forward Volatilities

The forward volatilities solve the following 'bootstrap' equations:

$$\begin{aligned} \text{capvol}_1^2 &= \sigma_0^2 && (23)\\ 2\text{capvol}_2^2 &= \sigma_0^2 + \sigma_1^2 && (24)\\ 3\text{capvol}_3^2 &= \sigma_0^2 + \sigma_1^2 + \sigma_2^2 && (25)\\ ... &= ... \\ N\text{capvol}_N^2 &= \sigma_0^2 + \sigma_1^2 + ... + \sigma_{N-1}^2 && (26) \end{aligned}$$

Computing Factor Loadings

1. Input constants $a_0(1), ..., a_{N-1}(1)$ [for convenience, assume $a_T(1) = (a_0(1))^{T+1}$, then only input $a_0(1)$.]

2. Compute the relative factor loadings for factor 2 using:

$$a_T(2) = (1 - a_T(1)^2)^{0.5}\ T = 0, 1, ..., N - 1 \tag{27}$$

3. Compute the absolute factor loadings for factor 1 and 2 using:

$$\begin{aligned} \sigma_T(1) &= a_T(1)\sigma_T,\ T = 0, 1, ..., N - 1 && (28)\\ \sigma_T(2) &= a_T(2)\sigma_T,\ T = 0, 1, ..., N - 1 && (29) \end{aligned}$$

Compute Covariances

Compute array of $\sigma_{\tau,T}(i)$, for factors $i = 1, 2$ and for $\tau = 0, 1, ..., N-1$ and $T = 0, 1, ..., N-1$, using

$$\sigma_{\tau,T}(i) = \sigma_\tau(i)\sigma_T(i) \tag{30}$$

The Evolution of the Forward rates

In the HSS method, the T-period forward rate at time t, in state r, s, [after r down-moves in factor 1 and s down-moves in factor 2] is given by

$$f_{t,t+T,r,s} = f_{t-1,t+T}[u_T(1)]^{t-r}[d_T(1)]^r[u_T(2)]^{t-s}[d_T(2)]^s \tag{31}$$

where

$$d_T(i) = \frac{2}{1 + e^{2\sigma_T(i)\sqrt{\delta}}}$$
$$u_T(i) = 2 - d_T(i),$$

for

$$t = 1, 2, ..., N$$
$$T = 0, 1, ..., N - t.$$

Here we have assumed that volatilities are time independent (i.e. they are dependent only on maturity T)

Forward Rate Drifts and HSS Probabilities

Let $m_{t,t+T}(i)$ denote the drift per period of the T-period forward rate at time t due to factor i. In general, the drift of the forward rate at time t is

$$m_{t,t+T,r,s}(i) = \delta\left[\frac{\delta f_{t,t+1,r,s}}{1+\delta f_{t,t+1,r,s}}\sigma_{0,T}(i) + \frac{\delta f_{t,t+2,r,s}}{1+\delta f_{t,t+2,r,s}}\sigma_{1,T}(i) + ... + \frac{\delta f_{t,t+T,r,s}}{1+\delta f_{t,t+T,r,s}}\sigma_{T,T}(i) - \frac{[\sigma_{T,T}(i)]^2}{2}\right]$$

and the probability of an up move is

$$\begin{aligned} q_{t,t+T,r,s}(i) &= [m_{t,t+T,r,s}(i) + (t-r)\ln u_{T+1}(i) + r\ln d_{T+1}(i) - (t-r)\ln u_T(i) - r\ln d_T(i) \\ &- \ln d_T(i)]/[\ln u_T(i) - \ln d_T(i)]. \end{aligned}$$

Bond Prices

First, compute the forward prices:

$$B_{t,t+T,t+T+1,r,s} = \frac{1}{1+\delta f_{t,t+T,r,s}}, T = 0, 1, 2, ...,$$

Then the long bond prices are given by:

$$B_{t,t+T+1,r,s} = B_{t,t+T,r,s} B_{t,t+T,t+T+1,r,s}$$

Caplet Prices

The European-style Caplet is priced using the equations:

$$caplet(t+T, t, r, s) = \max(f_{3,3} - k, 0) A\delta$$

where A Bermudan-style Caplet is priced using:

$$\begin{aligned} BM_3 &= \max(f_{3,3} - k, 0) A\delta \\ BM_2 &= \max[(f_{2,2} - k) A\delta, E_2(BM_3) B_{2,3}] \\ BM_1 &= \max[(f_{1,1} - k) A\delta, E_1(BM_2) B_{2,3}] \\ BM_0 &= E_0(BM_1) B_{0,1} \end{aligned}$$

10 Pricing Defaultable Bonds

General Approaches:

- Fundamental, structural models. These value bonds as options on an underlying value process. Example: Merton model
- Reduced form models. Assume an exogenous probability of default (hazard rate), plus a recovery rate. Examples: Duffie and Singleton, Jarrow and Turnbull

  Recovery rate assumptions

  1. Recovery of principal (face value) [RP]
  2. Recovery of treasury (present value) [RT]
  3. Recovery of market value [RMV]

Notation

- Let q_t be the risk-neutral probability of default over period t to $t+1$.
- Let $B_{t,T}$ be the value of a defaultable zero-coupon bond, with final maturity T.
- Let ψ_{t+1} be the dollar amount paid on the bond, in the event of a default.
- Let $b_{t,T}$ be the value of a risk-free zero-coupon bond.

Then taking the expectation under the risk-neutral measure over the joint distribution we can write:

$$B_{t,T} = [q_t E_t(\psi_{t+1}) + (1 - q_t) E_t(B_{t+1,T})] b_{t,t+1} \tag{32}$$

where $B_{t+1,T}$ is the value of the bond in the event of no default at time $t+1$. If the par value of the bond is 1 we then have

1. RP has $E_t(\psi_{t+1}) = \delta_t$
2. RT has $E_t(\psi_{t+1}) = \delta_t b_{t+1,T}$
3. RMV has
$$E_t(\psi_{t+1}) = \delta_t E_t(B_{t+1,T}) \tag{33}$$

Substituting (33) in (32), we have

$$B_{t,T} = [q_t \delta_t + (1 - q_t)] E_t(B_{t+1,T}) b_{t,t+1} \tag{34}$$

In a *LIBOR* model, we let

$$b_{t,t+1} = \frac{1}{1 + r_t h},$$

$$B_{t,T} = [q_t \delta_t + (1 - q_t)] E_t(B_{t+1,T}) \frac{1}{1 + r_t h}$$

In a similar manner to DS, we define a 'risk adjusted' rate R_t such that

$$B_{t,T} = \frac{1}{1 + R_t h} E_t(B_{t+1,T}) = [q_t \delta_t + (1 - q_t)] E_t(B_{t+1,T}) \frac{1}{1 + r_t h}$$

which implies that
$$R_t \approx r_t + q_t(1 - \delta_t)/h$$

10.1 A Credit Spread *LIBOR* Model

In Peterson and Stapleton (Pricing of Options on Credit-Sensitive Bonds) the London Interbank Offer Rate (*LIBOR*) is modelled as a lognormal diffusion process under the risk-neutral measure. Then, as in PSS, the second factor generating the term structure is the premium of the futures *LIBOR* over the spot *LIBOR*.

The second factor generating the premium is contemporaneously independent of the *LIBOR*. However, in order to guarantee that the no-arbitrage condition is satisfied, future outcomes of spot *LIBOR* are related to the current futures *LIBOR*. This creates a lag-dependency between spot *LIBOR* and the second factor. In addition ,the one-period credit-adjusted discount rate, appropriate for discounting credit-sensitive bonds, is given by the product of the one-period *LIBOR* and a correlated credit factor. We assume that this credit factor, being an adjustment to the short-term *LIBOR*, is independent of the futures premium.

This leads to the following set of equations:

We let (x_t, y_t, z_t) be a joint stochastic process for three variables representing the logarithm of the spot *LIBOR*, the logarithm of the futures-premium factor, and the logarithm of the credit premium factor.

We then have:

$$dx_t = \mu(x, y, t)dt + \sigma_x(t)dW_{1,t} \tag{35}$$
$$dy_t = \mu(y, t)dt + \sigma_y(t)dW_{2,t} \tag{36}$$
$$dz_t = \mu(z, t)dt + \sigma_z(t)dW_{3,t} \tag{37}$$

where $E\left(dW_{1,t}dW_{3,t}\right) = \rho$, $E\left(dW_{1,t}dW_{2,t}\right) = 0$, $E\left(dW_{2,t}dW_{3,t}\right) = 0$.

Here, the drift of the x_t variable, in equation (35), depends on the level of x_t and also on the level of y_t, the futures premium variable. Clearly, if the current futures is above the spot, then the spot is expected to increase. The mean drift of x_t thus allows us to reflect both mean reversion of the spot and the dependence of the future spot on the futures rate.

The drift of the y_t variable, in equation (36), also depends on the level of y_t, reflecting possible mean reversion in the futures premium factor. Note that equations (35) and (36) are identical to those in the two-factor risk-free bond model of Peterson, Stapleton and Subrahmanyam (2001).

The additional equation, equation (37), allows us to model a mean-reverting credit-risk factor. Also the correlation between the innovations $dW_{1,t}$ and

$dW_{3,t}$ enables us to reflect the possible correlation of the credit-risk premium and the short rate.

First, we assume, as in HSS, that x_t, y_t and z_t follow mean-reverting Ornstein-Uhlenbeck processes:

$$\begin{aligned} dx_t &= \kappa_1(a_1 - x_t)dt + y_{t-1} + \sigma_x(t)dW_{1,t} && (38) \\ dy_t &= \kappa_2(a_2 - y_t)dt + \sigma_y(t)dW_{2,t}, && (39) \\ dz_t &= \kappa_3(a_3 - z_t)dt + \sigma_z(t)dW_{3,t}, && (40) \end{aligned}$$

where $E\left(dW_{1,t}dW_{3,t}\right) = \rho dt,\ E\left(dW_{1,t}dW_{2,t}\right) = 0,\ E\left(dW_{2,t}dW_{3,t}\right) = 0.$ and where the variables mean revert at rates κ_j to a_j, for $j = x, y, z$.

As in Amin(1995), it is useful to re-write these correlated processes in the orthogonalized form:

$$\begin{aligned} dx_t &= \kappa_1(a_1 - x_t)dt + y_{t-1} + \sigma_x(t)dW_{1,t} && (41) \\ dy_t &= \kappa_2(a_2 - y_t)dt + \sigma_y(t)dW_{2,t} && (42) \\ dz_t &= \kappa_3(a_3 - z_t)dt + \rho\sigma_z(t)dW_{1,t} + \sqrt{1-\rho^2}\sigma_z(t)dW_{4,t}, && (43) \end{aligned}$$

where $E(dW_{1,t}dW_{4,t}) = 0$. Then, rearranging and substituting for $dW_{1,t}$ in (43), we can write

$$dz_t = \kappa_3(a_3 - z_t)dt - \beta_{x,z}\left[\kappa_1(a_1 - x_t)\right]dt + \beta_{x,z}dx_t + \sqrt{1-\rho^2}\sigma_z(t)dW_{4,t}.$$

In this trivariate system, y_t is an independent variable and x_t and z_t are dependent variables. The discrete form of the system can be written as follows: