

The Black Model and the Pricing of Options on Assets,  
Futures and Interest Rates

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## **Abstract**

### **The Black Model and the Pricing of Options**

We establish a general necessary and sufficient for the Black model to correctly price, to underprice, or to overprice European-style options. The condition for the Black model to hold is that the product of the asset price probability distribution and the pricing kernel is lognormal. This condition is applied to value stock options in particular cases where the pricing kernel exhibits declining or increasing elasticity, options on bonds where bond prices are lognormal, and options on interest rates where interest rates follow a Miltersen, Sandmann and Sondermann (1997)-Brace, Gatarek and Musiela (1997) process. We show that the Libor Market Model is just a special case of this class of Black models.

## 1 Introduction

The Black model is used in many different forms in option pricing. It is used in different forms for pricing and hedging stock options, index futures options, foreign exchange options and interest-rate options of various kinds. This is in spite of the fact that the assumptions normally used to establish the model: that the price of the underlying security follows a geometric Brownian motion and interest rates are non-stochastic are somewhat strong and hard to justify, and in some cases mutually inconsistent<sup>1</sup>. In this paper, we establish a necessary and sufficient condition for the Black model to correctly price European-style options. We then derive a new set of sufficient conditions under which the Black model holds. In the important special case of interest-rate options, we show conditions under which the Black model will correctly price options on bonds, interest rates and interest rate futures.

Although our main concern is to analyse the conditions for the Black model to hold in the case of individual options, a related question is when will the Black model price a series of options with different maturities. This is the question addressed by Brace, Gatarek and Musiela (1997) (BGM) and Miltersen, Sandmann and Sondermann (1997) (MSS). The MSS-BGM Libor Market Model is a model of the stochastic evolution of interest rates in which interest-rate options of all maturities are priced by the Black model. The conditions for the Black model to hold in this sense are more demanding because the interest rate plays a dual role: in the payoff function of the option and also as a discount factor, discounting option payoffs back from the maturity of the option to the valuation date.

The outline of this paper is as follows. In section 2 we derive the necessary and sufficient conditions for the Black model to hold for the forward price of an option. The product of the probability distribution function and the (asset specific) pricing kernel has to be lognormal. In section 3, by way of example we illustrate the point that neither the asset or the pricing kernel need be lognormal. In section 4, we extend the analysis by considering option valuation given stochastic interest rates. This allows us to analyse the Black model in the context of futures prices of options and also to find conditions for the Black model to price bond options and interest-rate options. These are analysed in section 5. In section 6, we look in detail at the Libor Market Model and show that the model is consistent with the Black model for the pricing of interest-rate caps and floors.

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<sup>1</sup>For example, Hull's proof of the Black model in the case of interest-rate futures options assumes that interest rates are non-stochastic.

## 2 Necessary and Sufficient Conditions for the Black Model

We consider the valuation of European-style options, with maturity date  $T$ , paying  $c(x_T)$ , which depend on an underlying security whose payoff is  $x_T$ . The security could be an asset with price  $x_T$ , or a foreign exchange rate, a futures price or an interest rate. We assume a no-arbitrage economy in which there exists a positive stochastic discount factor,  $\psi_{t,T}$ , that prices all claims. The spot price of the option, at time  $t$ , is

$$c_t = E[c(x_T)\psi_{t,T}]. \quad (1)$$

For convenience, we now define a variable  $\phi_{t,T}(x_T)$ :

$$\phi_{t,T}(x_T) \equiv E_t[\psi_{t,T}|x_T]/B_{t,T},$$

where  $B_{t,T}$  is the price of a risk-free zero-coupon bond paying \$1 at date  $T$ . We refer to this security specific pricing function  $\phi_{t,T}(x_T)$  simply as the pricing kernel.  $\phi_{t,T}(x_T)$  has the property  $E[\phi_{t,T}(x_T)] = 1$ . Using this pricing kernel, the forward price of the option is given by

$$F(c) = E_t[c(x_T)\phi_{t,T}(x_T)]. \quad (2)$$

If the option is a call option with strike price  $k$ , we denote its forward price as  $F[c(k)]$ . In this case, (2) becomes

$$F[c(k)] = E_t[\max(x_T - k, 0)\phi_{t,T}(x_T)]. \quad (3)$$

The Black model for such a call option is

$$F[c(k)] = F(x)N\left(\frac{\ln \frac{F(x)}{k} + \frac{\sigma_x^2(T-t)}{2}}{\sigma_x\sqrt{T-t}}\right) - kN\left(\frac{\ln \frac{F(x)}{k} - \frac{\sigma_x^2(T-t)}{2}}{\sigma_x\sqrt{T-t}}\right), \quad (4)$$

where  $F(x)$  is the forward price of  $x_T$ ,  $\sigma_x$  is the volatility of  $x_T$ , and  $T - t$  is the maturity of the option in years.

Now, let  $f(x_T)$  be the probability distribution function of  $x_T$ . We establish the following:

**Lemma 1** [*A Necessary and Sufficient Condition for the Black Model*]

*The Black model holds for the valuation of European-style options on  $x_T$  if and only if the risk adjusted probability distribution of  $x_T$ :*

$$\hat{f}(x_T) = f(x_T)\phi_{t,T}(x_T)$$

*is lognormal.*

Proof:

For sufficiency, see Poon and Stapleton, pp. 47-49. Necessity follows from the fact that all call options (for all  $k$ ) have to be priced by Black. This can only be true if the risk-neutral distribution is lognormal. [A more formal proof is required here]  $\square$

The significance of the lemma is that it shows that neither  $f(x_T)$  lognormal or  $\phi_{t,T}(x_T)$  lognormal is necessary for Black. Only the product of  $f(x_T)$  and  $\phi_{t,T}(x_T)$  has to be lognormal. Of course, in many treatments,  $f(x_T)$  lognormal is assumed, so that  $\phi_{t,T}(x_T)$  lognormal is necessary. This is the case in Brennan (1979) and Poon and Stapleton, ch. 3.

### 3 An Example: the Generalized Lognormal Distribution

In this section we simplify the notation, using  $x \equiv x_T$  and  $\phi(x) \equiv \phi_{t,T}(x_T)$ . By way of example, assume a distribution of the form

$$f(x) = a(q_1, q_2)g_2(x)e^{q_1 \ln x + q_2 k_2(x)}, \quad (5)$$

where  $g_2(x)$  is not dependent on  $q_1$  and  $q_2$  and  $q_2 \neq 0$  and where  $k(x)$  is declining in  $x$ . This generalization of the lognormal distribution was used by Franke, Huang and Stapleton (2005)(FHS) who show that a ‘two-dimensional risk-neutral valuation relationship’ holds in this case if the pricing kernel is of the form:

$$\phi(x) = b(\gamma_1, \gamma_2)e^{\gamma_1 \ln x + \gamma_2 k_2(x)}. \quad (6)$$

They also show that a ‘one-dimensional risk-neutral valuation relationship’ does not hold in this case. However, as noted above, this does not imply that the Black model does not hold as the following Proposition shows:

**Proposition 1** *Assume the probability distribution function is given by (5) and the pricing kernel has the form (6). Then the Black model underprices (correctly prices) [overprices] European-style options if and only if  $\gamma_2 < (=) [>] -q_2$*

Proof:

$$\begin{aligned} \hat{f}(x) &= \phi(x)f(x) \\ &= a(q_1, q_2)b(\gamma_1, \gamma_2)g_2(x)e^{(\gamma_1 + q_1) \ln x + (\gamma_2 + q_2)k_2(x)} \\ &= \hat{a}(q_1 + \gamma_1, q_2 + \gamma_2)g_2(x)e^{(\gamma_1 + q_1) \ln x + (\gamma_2 + q_2)k_2(x)}. \end{aligned} \quad (7)$$

If  $q_2 + \gamma_2 = 0$ ,  $\hat{f}(x)$  is lognormal and by Lemma 1, the Black model holds. If  $q_2 + \gamma_2 > 0$ , the risk-neutral density has declining elasticity and by FSS (1999) all options are underpriced by the Black model. Conversely, if  $q_2 + \gamma_2 < 0$ , the risk-neutral density has increasing elasticity and all options are overpriced by the Black model.

Note that, although the Black model holds, a one-dimensional risk-neutral valuation relationship does not hold in this case, since under risk neutrality,  $\hat{f}(x)$  is given by (5) and Black does not hold, if  $q_2 \neq 0$ .

### An Example

Suppose that  $k_2(x) = 1/x$ , which implies that the pricing kernel has declining elasticity. We have

$$\phi(x) = be^{\gamma_1 \ln x} e^{\frac{1}{x}},$$

assuming  $\gamma_2 = 1$ . Then the Black model holds if  $q_2 = -1$ , that is if

$$f(x) = ag_2(x)e^{q_1 \ln x} e^{-\frac{1}{x}}.$$

## 4 Option Pricing Under Stochastic Interest Rates

So far, we have analysed the conditions for the Black model using the ‘forward’ pricing kernel,  $\phi_{t,T}(x_T)$ . However, using  $\phi_{t,T}(x_T)$  tends to obscure two effects, that of risk aversion and of stochastic interest rates, on option pricing. Distinguishing these effects is particularly important for the pricing of interest-rate options. It is also crucial when considering the futures prices of options.

In order to analyse the valuation of options under stochastic interest rates, divide the interval from  $t$  to  $T$  into  $\delta$ -length sub-periods, with end periods  $t + 1, t + 2, \dots, T$ . Let the one-period Libor interest rate be  $i_\tau$ , for  $\tau = t, t + 1, t + 2, \dots, T - 1$ , with the one-period zero-coupon bond price

$$B_\tau = \frac{1}{1 + \delta i_\tau}.$$

From no-arbitrage, there exist pricing functions,  $\gamma_\tau$ , such that the price of an option at time  $\tau$  is

$$c_\tau = E_\tau[\gamma_{\tau+1} c_{\tau+1}] B_\tau$$

Using these pricing functions, the value of the option at time  $t$  is, from successive substitution,

$$c_t = E_t[\gamma_{t,T} g_{t,T} c(x_T)] B_{t,\tau}, \quad (8)$$

where

$$\gamma_{t,T} = \prod_{\tau=t}^{T-1} \gamma_{\tau+1}$$

and

$$g_{t,T} = \prod_{\tau=t}^{T-1} B_{\tau}/B_{t,T}.$$

Comparing the pricing equations (8) and (1) it follows that the stochastic discount factor in (1) is

$$\psi_{t,T} = \gamma_{t,T} g_{t,T} B_{t,T}. \quad (9)$$

(9) is merely a decomposition of the discount factor, which reveals the effects of risk aversion and stochastic interest rates. For example, if interest rates are non-stochastic,  $g_{t,T} B_{t,T} = 1$  and  $\psi_{t,T} = \gamma_{t,T}$ . Alternatively, if investors are (locally) risk neutral,  $\gamma_{t,T} = 1$  and  $\psi_{t,T} = g_{t,T} B_{t,T}$ .

Using the definition of the pricing kernel,

$$\phi_{t,T}(x_T) = E_t[\gamma_{t,T} g_{t,T} | x_T].$$

Hence, from Lemma 1, the Black model holds for the forward price of the option if and only if  $\hat{f}(x) = f(x) E_t[\gamma_{t,T} g_{t,T} | x_T]$  is lognormal. If interest rates are non-stochastic,  $g_{t,T} = 1$  and the condition becomes  $\hat{f}(x) = f(x) E_t[\gamma_{t,T} | x_T]$  is lognormal. On the other hand, if investors are (locally) risk neutral,  $\gamma_{t,T} = 1$  and the condition is  $\hat{f}(x) = f(x) E_t[g_{t,T} | x_T]$  is lognormal.

We can now distinguish the forward price and the futures price of an option. We have:

**Lemma 2** [*Option Prices under Stochastic Interest Rates*]

*The no-arbitrage forward price of an option paying  $c(x_T)$  under stochastic interest rates is*

$$F(c) = E_t[c(x_T) \gamma_{t,T} g_{t,T}].$$

*The no-arbitrage futures price of the option is*

$$H(c) = E_t[c(x_T) \gamma_{t,T}].$$

*Proof*

The forward price was established above. The *futures* price, assuming that the futures is marked to market at  $\delta$  intervals, is just the risk adjusted expectation of the option payoff, see for example Poon and Stapleton, ch 6.

The Black model for the futures price of a call option is

$$H[c(k)] = H(x)N\left(\frac{\ln \frac{H(x)}{k} + \frac{\sigma_x^2(T-t)}{2}}{\sigma_x\sqrt{T-t}}\right) - kN\left(\frac{\ln \frac{H(x)}{k} - \frac{\sigma_x^2(T-t)}{2}}{\sigma_x\sqrt{T-t}}\right), \quad (10)$$

where  $H(x)$  is the futures price of  $x_T$ . It now follows, by analogy with Lemma 1:

**Lemma 3** [*A Necessary and Sufficient Condition for the Black Model (Futures)*]

*The Black model holds for the futures value of European-style options on  $x_T$  if and only if the probability distribution of  $x_T$ :*

$$\hat{f}(x_T) = f(x_T)E_t[\gamma_{t,T}|x_T]$$

*is lognormal.*

Proof:

## 5 The Pricing of Interest-Rate Related Options

One of the most important applications of the Black model is in the area of interest-rate options. The Black model has been employed to price bond options, swaptions, Eurodollar futures options, and interest-rate caps and floors. In this section we apply the results derived in the previous section to find the conditions under which the Black model applies to these options. We work under the assumption of (local) risk neutrality.

We have:



**Proposition 2** [*The Black model for Interest-Rate Options, under (local) Risk-Neutrality*]

Consider the pricing of options paying  $c(x_T)$  assuming  $\gamma_{t,T} = 1$ . Then

1. Bond Options:  $x_T = B_{T,T+n}^c$

- (a) The Black model holds for the futures price of the option iff the bond price distribution  $f(B_{T,T+n}^c)$  is lognormal.
- (b) The Black model holds for the forward price of the option iff the bond price distribution  $\hat{f}(B_{T,T+n}^c) = f(B_{T,T+n}^c)E_t[g_{t,T}|x_T]$  is lognormal.

2. Interest-Rate Options:  $x_T = i_T$

- (a) The Black model holds for the futures price of the option iff the interest-rate distribution  $f(i_T)$  is lognormal.
- (b) The Black model holds for the forward price of the option iff the interest-rate distribution  $\hat{f}(i_T) = f(i_T)E_t[g_{t,T}|x_T]$  is lognormal.

Proof:

1. (a) From Lemma 2 the futures price of the option is

$$H(c) = E_t[c(x_T)\gamma_{t,T}].$$

With  $x_T = B_{T,T+n}^c$  and  $\gamma_{t,T} = 1$ , the result follows using Lemma 3.

- (b) Similarly, again using Lemma 2, the forward price of the option is

$$F(c) = E_t[c(x_T)\gamma_{t,T}g_{t,T}].$$

With  $x_T = B_{T,T+n}^c$  and  $\gamma_{t,T} = 1$ , the result follows using Lemma 1.

2. (a) From Lemma 2 the futures price of the option is

$$H(c) = E_t[c(x_T)\gamma_{t,T}].$$

With  $x_T = i_T$  and  $\gamma_{t,T} = 1$ , the result follows using Lemma 3.

- (b) Similarly, again using Lemma 2, the forward price of the option is

$$F(c) = E_t[c(x_T)\gamma_{t,T}g_{t,T}].$$

With  $x_T = i_T$  and  $\gamma_{t,T} = 1$ , the result follows using Lemma 1.

## Applications

### 1. Options on Eurodollar Futures

These are traded in London on a futures (marked-to-market) basis. Although they are American-style, their value can be approximated by assuming that they pay off on a European-style basis. Assuming also that the maturity of the underlying futures is the same as the option maturity, the payoff on a put option is

$$A \max(K - H_{T,T}, 0) = A' \max(i_T - k, 0)$$

Here, the Black model applies, if  $f(i_T)$  is lognormal. Note that this is the case in the Black-Karasinski model.

### 2. Interest-Rate Caps and Floors: Bond Prices Lognormal

An interest-rate cap (floor) can be decomposed into a series of caplets (floorlets) each of which is equivalent to a put option at par on a particular coupon bond <sup>2</sup>. The payoff on a caplet is

$$caplet_T = \max\left(1 - \frac{1 + \delta k}{1 + \delta i_T}\right)$$

In this case, the Black model can be applied if the bond price

$$B_{T,T+\delta}^c = \frac{1 + \delta k}{1 + \delta i_T}$$

is lognormal and the discount factor  $g_{t,T}$  is also lognormal. Note that this is the case in the Ho-Lee and Hull-White models.

### 3. Interest-Rate Caps and Floors: Interest Rates Lognormal

Let  $x_T = i_T$ . The Black model (forward version) holds for the valuation of a caplet if  $\hat{f}(i_T)$  is lognormal. This is the case in the MSS-BGM model. Note this must be the case, but it needs a proof.

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<sup>2</sup>See Stapleton and Subrahmanyam (1991) for a detailed analysis of caps and floors. MSS (1997) also use this characterization of these products.

## 6 The Black Model in a Multi-Period Economy

**Lemma 4** [*The Black Model for a Series of Options*]

Consider an  $n$ -period economy where each period is length  $\delta$ . Assume local risk neutrality. Suppose there exist options on an underlying security with payoffs  $c(x_{t+j\delta})$  for  $j = 1, 2, \dots, n$ . Then, the Black model holds:

1. For the forward price of  $c(x_{t+\delta})$  iff  $f(i_{t+\delta})$  is lognormal.
2. For the forward price of  $c(x_{t+2\delta})$  iff  $f(i_{t+\delta})E_t(g_{t,t+2\delta}|x_T)$  is lognormal.
3. For the forward price of  $c(x_{t+j\delta})$  iff  $f(i_{t+\delta})E_t(g_{t,t+j\delta}|x_T)$  is lognormal.

Proof

Follows directly from Lemma 1.

**Proposition 3** [*The Libor Market Model in a Two-Period Economy*]

Let  $x_{t+\delta} = i_{t+\delta}$ , and  $x_{t+2\delta} = i_{t+2\delta}$ . Then the Black (forward) model holds simultaneously for options paying  $c(i_{t+\delta})$  and  $c(i_{t+2\delta})$  iff both

1.  $f(i_{t+\delta})$  is lognormal, and
- 2.

$$\hat{f}(i_{t+2\delta}) = f(i_{t+2\delta}) \left[ \frac{1}{1 + \delta i_{t+\delta}} \frac{B_{t,t+\delta}}{B_{t,t+\delta}} \right]$$

is lognormal.

Proof:

1. follows directly from Lemma 4 above. The term in brackets is the stochastic discount factor,  $g_{t,t+2\delta}$ . In the case of an interest-rate process the value of this is known given  $i_T$ , hence 2. follows.

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