Forward-Price Processes and the Risk-Neutral Pricing of Options

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Abstract

This paper investigates the preference restrictions which underlie the Black-Scholes (lognormal), Brennan (normal), and Rubinstein (generalized lognormal) option pricing models. It also introduces a fourth option pricing model for assets which have negatively skewed returns. It establishes new sufficient conditions for the models to hold in a multi-asset economy. First, assuming that expectations of an asset price follow a lognormal diffusion, we derive the Black-Scholes model in an economy where the representative agent has an extended power utility function of wealth. We then establish the Brennan model assuming that expectations follow a normal diffusion in an economy where the representative agent again has an extended power utility function. Then we assume that expectations follow a displaced diffusion and derive conditions under which the Rubinstein model holds. Finally we assume that an asset has a negatively skewed lognormal distribution as in the bond option model of Stapleton and Subrahmanyam (1993) and again derive conditions for risk-neutral pricing of options.

1 Introduction

Option pricing models normally start by assuming that the price of an asset follows a given stochastic process. In the Black-Scholes model, the spot price of the asset is assumed to follow a geometric Brownian motion. In the Black model, either the forward price or the futures price is assumed to follow a geometric Brownian motion. Further examples are Brennan (1979)'s model which can be derived by assuming that the asset follows an arithmetic Brownian motion and Rubinstein (1983)'s displaced diffusion model. However, as emphasised in the work of Bick (1987, 1990), the price process for an asset can be derived endogenously, given more basic assumptions about the production process and utility functions. It then transpires that the conditions under which the price process is a geometric Brownian motion, or one of the other above processes, are quite restrictive. Bick (1987), for example, shows that constant proportional risk aversion (CPRA) is a necessary and sufficient condition for the price process to follow a geometric Brownian motion, if the production process follows such a process, albeit in a one-asset model. Also, if these conditions do not hold, then the consequences for option pricing are immediate. As shown, for example, by Franke, Stapleton and Subrahmanyam (1999), if investors have declining risk aversion, all options will be systematically underpriced by the Black-Scholes model. The study of economies in which the Black-Scholes and other 'risk-neutral' option pricing models hold is therefore of some importance.

The literature discussing conditions for endogenous price processes, and the related literature on the existence of risk-neutral-valuation relationships for option pricing in single-period economies, is reviewed in section 2. The overwhelming impression gained from this literature is that the necessary and sufficient conditions for an asset price to follow geometric Brownian motion are that the representative investor has CPRA utility (see for example Bick (1987), Stapleton and Subrahmanyam (1990), He and Leland (1995), and Brennan (1979)). If this were true, the Black-Scholes model would only strictly apply in quite restrictive conditions. However, this is not true. In a multi-asset economy, the asset on which the option is written and the market portfolio can, and most likely will, follow different stochastic processes.[1] In this paper, we find alternative conditions for the Black-Scholes model to hold, by allowing the market portfolio to follow a different stochastic process from

[1] In the Brennan-Rubinstein discrete time option pricing model it is assumed that the stock and the market folows a joint lognormal process. While this is possible for a single asset, it is not possible for all assets, since the sum of lognormals is not lognormal. Also, evidence from option implied volatilities suggests that individual assets and the market portfolio follow different processes.

that of the individual asset on which the option is written.

In this paper, we break the link between the information process followed by the individual asset and the market portfolio. As well as allowing us to find alternative economies which support the Blkack-Scholes model, it also allows us to consider conditions for the risk-neutral valuation of options on different assets, with different distributions, within the same economy. For example, assume there are four different assets. One has an information process which follows standard geometric Brownian motion. A second follows a displaced diffusion process as in Rubinstein (1983). A third asset follows an arithmetic Brownian motion. A fourth has a negatively skewed distribution, where one minus the price is lognormally distributed. All these cases are quite relevant. As discussed in Rubinstein (1983) and Camara (1999), the displaced diffusion process is relevant when valuing options (for example the debt or equity) on a firm with cash assets. The cash puts a lower non-zero limit on the value of the firm. Again, as discussed in Brennan (1979), the arithmetic Brownian motion may well be more appropriate for valuing options on physical quantities or on profits, which can become negative. Also, as proposed by Stapleton and Subrahmanyam (1993), it may be reasonable to model a zero-coupon bond price as a negatively skewed lognormal variable. Given these four different asset information processes, we ask the question as to when the forward price of each of the assets will also follow the same process as that followed by its information process.

This paper is organised as follows. In section 2 we review the most relevant articles in the literature, concluding that the overwhelming majority of the research concentrates either on single-asset economies or on the special case where the asset and the market follow a similar process. Section 3 then looks at the case of geometric Brownian motions. It shows a range of economies in which the price of an asset follows a geometric Brownian motion and the Black-Scholes model holds. These include the cases where the aggregate wealth follows a displaced diffusion process and utility is HARA and the case where the aggregate wealth follows a Brownian motion and utility is CARA. In section 4 we then assume that assets in the same economy can follow any of four different processes and establish conditions under which the forward prices of the assets follow similar processes. This also establishes common risk-neutral-valuation relationships for the valuation of options on the various assets. Section 5 presents the conclusions of the paper.

2 Previous Literature

Bick (1987) considered economies that are consistent with the Black-Scholes model. He assumes a production process for a single consumption good that follows a geometric Brownian motion and shows that the endogenously derived stock price process follows a geometric Brownian motion, if and only if the investor has CPRA. Also, Bick showed that if the Black-Scholes model is to hold for options on the asset, then a necessary and sufficient condition is that the investor has a utility function that exhibits constant proportional risk aversion. He and Leland (1995) use a rather different set up but again conclude that, in a single asset economy, the Black-Scholes model holds if the representative investor has a CPRA utility function.

Stapleton and Subrahmanyam (1990) consider the following related question. When is a random walk in the expectation of a cash flow translated into a random walk in the value of a cash flow? They consider both geometric random walks and arithmetic random walks. They conclude that a geometric random walk in the expectation of an asset becomes a geometric random walk in the value of the asset, if and only if the pricing kernel is a power function of the asset return. They then consider economies where the asset return and the market portfolio follow a joint geometric random walk process and show that the price process for the aset will be a geometric random walk if and only if the representative investor has CPRA preferences.

Franke, Stapleton and Subrahmanyam (1999) show that the crucial property that determines whether prices follow random walks is the elasticity of the pricing kernel. They show that if the expectation of a cash flow follows a geometric diffusion process then the value of the cash flow also follows a geometric diffusion process, if the pricing kernel has constant elasticity. Their main purpose is to show that if the pricing kernel exhibits declining elasticity then the value process has excess volatility and autocorrelation. This in turn leads to the overpricing of options compared to the Black-Scholes model.

Bick (1990) considers the more general question of the 'viability' of a stochastic process for the price of an asset. Again he restricts the analysis to a single asset economy. His paper studies the following question: which assumptions on the price process of the market portfolio "make sense"? For the HARA-class of utility functions Bick indentifies the class of diffusion processes that are viable, given these utility functions. The present paper follows Bick in investigating the whole set of HARA functions.

The question as to whether we get geometric or arithmetic random walks in cash flows is closely related to the conditions for RNVRs to exist in a dynamic incomplete discrete time economy. Merton (1973) and Rubinstein (1976) first found that the Black-Scholes model holds in a discrete economy where investors have CPRA utility and where the asset price and the market portfolio are joint-lognormally distributed. Brennan (1979) then showed that CPRA was a necessary condition if the Black-Scholes model was to hold for options on the market portfolio. Brennan also showed that if the asset and the market portfolio are joint-normally distributed, as in the Sharpe-Lintner CAPM, then a RNVR for options (analogous to the Black-Scholes model) holds if and only if investors have exponential utility.

A significant restriction in most of the above literature is that it covers either the price process for the asset in a single asset economy, or assumes that the process followed by the asset and the market portfolio is the same.[2] In this paper we employ a similar idea. We allow the joint process for the asset and the aggregate wealth to follow one of a set of joint processes. By relaxing the unnecessary restriction that the asset and the market follow the same type of process we are also able to consider the price processes for a range of possible assets in the same economy. We are then able to apply any of Bick's class of viable processes for aggregate wealth, given HARA utility in order to price options on assets with a range of possible distributions.

[2] An exception is the recent paper by Camara (1999) who finds conditions for the risk-neutral valuation of options in an economy where the asset and the market follow a bivariate lognormal-normal process.

3 Conditions for a Lognormal Marginal Utility of Wealth

We work with the following generalized definition of bivariate normality:

Definition 1 *(The generalized Bivariate Normal Distribution) Two variables Y and Z are generalized bivariate-normally distributed if and only if there exist functions $g(Y)$ and $h(Z)$ that are bivariate normally distributed.*

For example, if $g(Y)$ =ln(Y), and $h(Z) = Z$, Y and Z are said to be bivariate lognormal-normal. As a second example, if $g(Y)$ =ln$(a+Y)$, and $h(Z)$ =ln(Z), Y and Z are bivariate displaced lognormal-lognormal.

Definition 2 *(The HARA marginal utility function) The marginal utility functions of the HARA (hyperbolic absolute risk aversion) or LRT (linear risk tolerance) class are given by the following equation:*

$$U^{'}(W_T) = \left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi} \tag{1}$$

where W_T denotes terminal wealth, and α, φ and θ are parameters, such that $\frac{\alpha W_T}{1-\varphi} + \theta > 0$.

It is well known that if $\alpha > 0$ and $\varphi < 0$ then $\theta < 0$ implies DPRA, $\theta = 0$ implies CPRA, and $\theta > 0$ implies IPRA. If $\alpha > 0$, $\varphi \to -\infty$, $\theta = 1$ then utility displays CARA.

We now prove a result which is fundamental for the analysis of endogenous price processes. In equilibrium economies, the pricing kernel depends upon on the marginal utility function of the representative investor. The following lemma establishes conditions under which the marginal utility function is lognormal. we have:

Lemma 1 *Assume that $g(W_T) \sim N(\mu_w, \sigma_w^2)$, where $g(W_T)$ is a function of the terminal wealth, W_T.*

The following conditions are sufficient for $U'(W_t)$ to be lognormal:

1. *Terminal wealth is normally distributed and CARA utility:*

$$g(W_T) = W_T \text{ and } \varphi \to -\infty,\ \theta = 1,\ \alpha > 0$$

2. *Terminal wealth is lognormally-distributed and CPRA utility:*

$$g(W_T) = \ln(W_T) \text{ and } \varphi < 0,\ \theta = 0,\ \alpha > 0$$

3. *Terminal wealth is displaced-lognormally distributed wealth and HARA utility:*

$$g(W_T) = \ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right] \text{ and } \varphi < 0,\ \alpha > 0$$

Proof

1. $g(W_T) = W_T$ means that $W_T \sim N(\mu_w, \sigma_w^2)$. $\varphi \to -\infty,\ \theta = 1,\ \alpha > 0$ implies that

   $$U'(W_T) = e^{-\alpha W_T}.$$

   Hence

   $$\ln[U'(W_T)] = -\alpha W_T,$$

   which is normal since W_T is normal.

2. $g(W_T) = \ln(W_T)$ means that $\ln(W_T) \sim N(\mu_w, \sigma_w^2)$. $\theta = 0$ implies that

   $$U'(W_T) = \left(\frac{\alpha W_T}{1-\varphi}\right)^{\varphi}$$

   which is lognormal since

   $$\ln[U'(W_T)] = \varphi\left[\ln\left(\frac{\alpha}{1-\varphi}\right) + \ln(W_T)\right]$$

   which is normal, since $\ln(W_T)$ is normal.

3. $g(W_T) = \ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right]$ means that $\ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right] \sim N(\mu_w, \sigma_w^2)$. Marginal utility

   $$U'(W_T) = \left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}$$

   is lognormal since

   $$\ln[U'(W_T)] = \varphi\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)\right]$$

   which is normal, since

   $$\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)$$

   is normal.

4 Geometric Random Walk for the Asset Price

In this section, we discuss the most important case in the literature, where the asset price follows a geometric random walk. A sufficient condition for the Black-Scholes theorem to hold is that the forward price of the asset follows such a process. Here we assume that the conditional expectation of the terminal price follows a lognormal geometric random walk and derive conditions under which the endogenously derived forward price follows a similar process.

Theorem 1 *(GRW for the forward price)*

Assume that a representative agent exists with a utility function of the HARA family. Let the conditional expectation at time t of the asset price X_T follow a lognormal geometric random walk process, so that:

$$\ln[E_t(X_T)] \sim N[\mu_x, \sigma_x], \qquad t \leq T$$

Then the forward price $F_t(X_T)$ follows a lognormal geometric random walk if either:

1. *$\varphi \to -\infty$, $\theta = 1$, and $\alpha > 0$; that is preferences are characterised by a negative exponential utility function, and wealth W_T and X_T are joint normal-lognormal.*

2. *$\alpha > 0$, $\varphi < 0$, and $\theta = 0$, that is preferences are CPRA, and wealth W_T and X_T are joint lognormal.*

3. *$\alpha > 0$, $\varphi < 0$, and $W_T > -\frac{(1-\varphi)\theta}{\alpha}$; that is preferences are characterised by an extended power utility function, and wealth W_T and X_T are joint displaced lognormal-lognormal.*

Proof

It has been established in the literature ,that the forward price of X_T, denoted $F_{t,T}(X_T)$, follows a geometric random walk, if and only if the pricing kernel $\phi_{t,T}$ defined by $F_t(X_T) = E_t(X_T\phi_{t,T})$, where $E_t(\phi_{t,T}) = 1$, has the property

$$\psi_{t,T} \equiv E_t(\phi_{t,T}|X_T) = AX_T^{\beta}$$

for constants A and β. In other words, the asset specific pricing kernel $\psi_{t,T}$, is a power function of the cash flow X_T. [3] We will show that any of the conditions 1), 2), and 3) above are sufficient for this condition to hold.

1. $g(W_T) = W_T$ means that $W_T \sim N(\mu_w, \sigma_w^2)$. $\varphi \to -\infty$, $\theta = 1$, $\alpha > 0$ implies from Lemma 1 that

$$U'(W_T) = e^{-\alpha W_T}.$$

Hence

$$\ln[U'(W_T)] = -\alpha W_T,$$

which is normal since W_T is normal. Also, since W_T is normal,

$$E_t[U^{'}(W_T)] = exp\left\{-\alpha\mu_w + \frac{\alpha^2}{2}\sigma_w^2\right\} \tag{2}$$

Since W_T and $\ln X_T$ are joint normal we can write the linear regression:

$$-\alpha W_T = a + b\ln(X_T) + \epsilon \tag{3}$$

where $\ln X_T$ is independent of ϵ. From (3) it follows, taking variances

$$\alpha^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \alpha^2\sigma_w^2 - b^2\sigma_x^2 \tag{4}$$

Also, from (3) it follows, taking expectations that

$$-\alpha\mu_w = a + b\mu_x$$

and hence the conditional expectation

$$E_t[\ln(U^{'}(W_T))|X_T] = a + b\ln(X_T) = -\alpha\mu_w - b\mu_x + b\ln(X_T), \tag{5}$$

where A is a constant. Using equations (4) and and (5), it follows that

$$\begin{aligned}\psi_{t,T} &= E_t\left[U^{'}(W_T) \mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\ &= \frac{exp\left[-\alpha\mu_w - b\mu_x + b\ln(X_T) + (\alpha^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[-\alpha\mu_w + (\alpha^2\sigma_w^2)/2\right]} \\ &= AX_T^b.\end{aligned}$$

[3]This follows, for example, from a special case of Stapleton and Subrahmanyam (1990)

Hence, the asset specific pricing kernel is a power function of the cash flow X_T and the price of X_T therefore follows a geometric random walk. □

2. Cases 2 and 3

$$g(W_T) = \ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right]$$

with $\theta = 0$, in case 2, implies by Lemma 1 that

$$U'(W_T) = \left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}$$

is lognormal, and has logarithmic mean

$$E_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right]$$

and logarithmic variance

$$Var_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi^2\sigma_w^2$$

It follows that

$$E_t\left[U^{'}(W_T)\right] = exp\left\{\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] + \frac{1}{2}\varphi^2\sigma_w^2\right\}. \tag{6}$$

Since W_T and X_T are joint displaced lognormal-lognormal, $\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)$ and $\ln(X_T)$ are joint normal, and hence we can write the linear regression:

$$\varphi\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right) = a + b\ln(X_T) + \epsilon \tag{7}$$

where $\ln(X_T)$ is independent of ϵ. From (7) it follows, taking variances

$$\varphi^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \varphi^2\sigma_w^2 - b^2\sigma_x^2. \tag{8}$$

Also, from (7) it follows, taking expectations that

$$\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] = a + b\mu_x$$

and hence the conditional expectation

$$E_t[\ln(U^{'}(W_T))|X_T] = a + b\ln(X_T) = \varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ln(X_T). \tag{9}$$

Using equations (8) and and (9), it follows that

$$\begin{aligned}\psi_{t,T} &= E_t\left[U^{'}(W_T)\mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\ &= \frac{exp\left[\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ln X_T + (\varphi^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] + (\varphi^2\sigma_w^2)/2\right]} \\ &= AX_T^b.\end{aligned}$$

Hence, the asset specific pricing kernel is a power function of the cash flow X_T and the price of X_T therefore follows a geometric random walk. In particular, when in case 2 $\theta = 0$ and wealth W_T and X_T are joint lognormal, the price of X_T again follows a geometric random walk.

□

Theorem 1 is interesting because it provides a whole range of economies in which the Black-Scholes model prices options on lognormally distributed assets. Whereas Bick (1989), Leland (1995), Ritchen and Mathur (1995), Franke, Stapleton and Subrahmanyam (1998), Brennan (1979), Rubinstein (1976) all derive the Black-Scholes model in economies where aggregate wealth is lognormally distributed and the representative agent has CPRA preferences, Theorem 1 assumes the more general HARA class of utility functions. It provides, for example several special cases where the model holds. We summarize important examples in the following corollary:

Corollary 1 *Assume that the conditional expectation of the individual asset follows a geometric Brownian motion and either:*

1. *Aggregate wealth is lognormal and the utility of the representative agent exhibits CPRA.*
2. *Aggregate wealth is normal and the utility of the representative agent exhibits CARA.*
3. *Aggregate wealth is displaced lognormal and the representative agent has an extended power utility function that displays either DPRA, CPRA, or IPRA.*

Then the Black-Scholes model holds for the pricing of European-style options on indiviudual assets.

Proof: A geometric Brownian motian is a lognormal geometric random walk in continuous time. Hence Theorem 1 applies and the asset forward price follows a geometric Brownian motion. This condition is sufficient, given frictionless markets, for the Black-Scholes model to hold.□

5 Displaced-Geometric Random Walks and Negatively-Skewed-Geometric Random walks: Sufficient Conditions for Price Processes

In this section we generalize previous results in the literature to two cases, where the conditional expectation of a cash flow X_T follows a displaced-geometric random walk (DGRW), and negatively-skewed-geometric random walk respectively. These results will then be used in the following section to establish sufficient conditions for the forward prices of assets to follow similar processes.

Lemma 2 *Assume that the asset price X_T follows a displaced geometric random walk. Then the forward price of X_T follows a displaced geometric random walk if the asset specific pricing kernel, $\psi_{t,T} = E_t(\phi_{t,T}|X_T)$ has the form: $\psi_{t,T} = a(X_T - \beta)^\alpha$.*

Proof First, note that since the pricing kernel, ϕ, has an expectation of 1, the asset specific pricing kernel $\psi_{t,T} = E_t(\phi_{t,T}|X_T)$, also has an an expectation of 1. Now suppose that $\psi_{t,T} = a(X_T - \beta)^\alpha$.

Since $X_T = \beta + [E_t(X_T) - \beta]y_{t,T}$,

$$E_t(X_T) = \beta + E_t(y_{t,T})(E_t(X_T) - \beta).$$

Hence, we can write:

$$\frac{X_T}{E_t(X_T) - \beta} = \frac{\beta + (E_t(X_T) - \beta)y_{t,T}}{E_t(y_{t,T})(E_t(X_T) - \beta)}.$$

Since $E_t(\psi_{t,T}) = aE_t(X_T - \beta)^\alpha = aE_t[(E_t(X_T) - \beta)^\alpha y_{t,T}^\alpha] = 1$ it follows that:

$$a = \frac{1}{(E_t(X_T) - \beta)^\alpha E_t(y_{t,T}^\alpha)}.$$

Hence,

$$\begin{aligned}\psi_{t,T} &= \frac{(X_T-\beta)^\alpha}{(E_t(X_T)-\beta)^\alpha E_t(y_{t,T}^\alpha)} \\ &= \frac{y_{t,T}^\alpha}{E_t(y_{t,T}^\alpha)}.\end{aligned}$$

It follows that:

$$Cov_t\left[\frac{X_T}{E_t(X_T)-\beta},\psi_{t,T}\right] = Cov_t\left[\frac{y_{t,T}}{E_t(y_{t,T})},\frac{y_{t,T}^\alpha}{E_t(y_{t,t}^\alpha)}\right]$$

which is nonstochastic. Given that $Cov_t\left[\frac{X_T}{E_t(X_T)-\beta},\psi_{t,T}\right]$ is non-stochastic we now show that the forward price of X_T follows a displaced geometric random walk. First, we define ϵ_t by the relationship

$$X_T-\beta = [F_t(X_T)-\beta]\epsilon_{t,T}.$$

Then, since the forward price is given by

$$F_t(X_T) = E_t(X_T\psi_{t,T}),$$

we can write:

$$\epsilon_{t,T} = \frac{X_T-\beta}{E_t(X_T)+Cov_t(X_T,\psi_T)-\beta}.$$

Given that X_T follows a displaced-geometric random walk, $X_T-\beta = (E_t(X_T)-\beta)y_{t,T}$, where $y_{t,T}$ is independent of t. Then $\epsilon_{t,T}$ can be written:

$$\epsilon_{t,T} = \frac{y_{t,T}}{1+Cov_t\left[\frac{X_T}{E_t(X_T)-\beta},\psi_{t,T}\right]}$$

Since $y_{t,T}$ is independent of $E_t(X_T)$ and $Cov_t[\frac{X_T}{E_t(X_T)-\beta},\psi_T]$ is non-stochastic, then $\epsilon_{t,T}$ is independent of the state of the world at t. Hence, the forward price of X_T follows a displaced-geometric random walk. □

We now extend our results to the case of negatively-skewed-geometric random walks. A cash flow follows such a process if $\beta-X_T = (\beta-X_t)y_{t,T}$ and the logarithm of $y_{t,T}$ is a normally distributed noise, independent $E_t(X_T)$.

Lemma 3 *Assume that the asset price X_T follows a negatively-skewed-geometric random walk. Then the forward price $F_t(X_T)$ follows a negatively-skewed-geometric random walk if $\psi_{t,T} = a(\beta-X_T)^\alpha$.*

Suppose that $\psi_{t,T} = a(\beta - X_T)^\alpha$.

Since $X_T = \beta + [\beta - E_t(X_T)]y_{t,T}$,

$$E_t(X_T) = \beta + E_t(y_{t,T})(\beta - E_t(X_T))$$

we can write:

$$\frac{X_T}{\beta - E_t(X_T)} = \frac{\beta - (\beta - E_t(X_T))y_{t,T}}{E_t(y_{t,T})(\beta - E_t(X_T))}.$$

Since $E_t(\psi_{t,T}) = aE_t[(\beta - E_t(X_T))^\alpha y_{t,T}^\alpha] = 1$, it follows that:

$$a = \frac{1}{(\beta - E_t(X_T))^\alpha E_t(y_{t,T}^\alpha)}.$$

Hence the asset specific pricing kernel can be written in the following form:

$$\psi_{t,T} = \frac{y_{t,T}^\alpha}{E_t(y_{t,T}^\alpha)}.$$

It follows that:

$$Cov_t\left[\frac{X_T}{\beta - E_t(X_T)}, \psi_{t,T}\right] = Cov_t\left[\frac{y_{t,T}}{E_t(y_{t,T})}, \frac{y_{t,T}^\alpha}{E_t(y_{t,t}^\alpha)}\right],$$

which is nonstochastic.

Given that $Cov_t\left[\frac{X_T}{\beta - E_t(X_T)}, \psi_{t,T}\right]$ is non-stochastic we now show that the forward price of X_T follows a negatively-skewed geometric random walk. First, we define $\epsilon_{t,T}$ by the relationship:

$$\beta - X_T = [\beta - F_t(X_T)]\epsilon_{t,T}.$$

Then, since the forward price is given by $F_t(X_T) = E_t(X_T\psi_{t,T})$, we can write:

$$\epsilon_{t,T} = \frac{\beta - X_T}{\beta - E_t(X_T) - Cov_t(X_T, \psi_{t,T})}.$$

Given that X_T follows a negatively-skewed geometric random walk, $\beta - X_T = (\beta - E_t(X_T))y_{t,T}$, where $y_{t,T}$ is independent of t. Then $\epsilon_{t,T}$ can be written:

$$\epsilon_{t,T} = \frac{y_{t,T}}{1 - Cov_t\left[\frac{X_T}{\beta - E_t(X_T)}, \psi_{t,T}\right]}.$$

Since $y_{t,T}$ is independent of $E_t(X_T)$ and $Cov_t[\frac{X_T}{\beta - E_t(X_T)}, \psi_T]$ is non-stochastic, then $\epsilon_{t,T}$ is independent of the state of the world at t. Hence, the forward price of X_T follows a negatively-skewed geometric random walk. □

6 An Economy with Assets Following Different Processes

In this section we consider an economy in which the conditional expectation of the terminal asset price follows any of the four different processes:

1. A geometric random walk.
2. An arithmetic random walk.
3. A displaced geometric random walk.
4. A negatively skewed geometric random walk.

We ask the question: under what conditions are each of these processes preserved in the case of the forward price of the asset? We are then able to establish sufficient conditions for risk-neutral-valuation relationships to apply for each type of asset in the same economy.

From Theorem 1 we know three sets of sufficient conditions for the forward price of an asset, who's expectation follows a GRW, to follow a GRW. We now show that the same conditions suffice in the case of an asset following an ARW, a DGRW, and a NSGRW. We begin by establishing the result in the case of an asset whose conditional expectation follows an arithmetic, normal random walk. We have:

Theorem 2 *(ARW for the forward price)*

Assume that a representative agent exists with a utility function of the HARA family. Let the conditional expectation at time t of the asset price X_T follow a normal arithmetic random walk process, so that:

$$E_t(X_T) \sim N[\mu_x, \sigma_x], \qquad t \leq T$$

Then the forward price $F_t(X_T)$ follows a normal arithmetic random walk if either:

1. *$\varphi \to -\infty$, $\theta = 1$, and $\alpha > 0$; that is preferences are characterised by a negative exponential utility function, and wealth W_T and X_T are joint normal.*
2. *$\alpha > 0, \varphi < 0 and \theta = 0$, that is preferences are CPRA, and wealth W_T and X_T are joint lognormal-normal.*

3. $\alpha > 0$, $\varphi < 0$, *and* $W_T > -\frac{(1-\varphi)\theta}{\alpha}$; *that is preferences are characterised by an extended power utility function, and wealth* W_T *and* X_T *are joint displaced lognormal-normal.*

Proof

The proof is similar to the proof of Theorem 1. From a special case of Stapleton and Subrahmanyam (1990), the forward price of X_T, denoted $F_{t,T}(X_T)$ follows a arithmetic random walk, if and only if the pricing kernel $\phi_{t,T}$ defined by $F_t(X_T) = E_t(X_T\phi_{t,T})$ has the property

$$\psi_{t,T} \equiv E_t(\phi_{t,T}|X_T) = Ae^{bX_T}$$

for constants A and b. In other words, the asset specific pricing kernel $\psi_{t,T}$, is an exponential function of the cash flow X_T. We will show that any of the conditions 1), 2), and 3) above are sufficient for this condition to hold.

1. $g(W_T) = W_T$ means that $W_T \sim N(\mu_w, \sigma_w^2)$. $\varphi \to -\infty$, $\theta = 1$, $\alpha > 0$ implies from Lemma 1 that

$$U'(W_T) = e^{-\alpha W_T}.$$

Hence

$$\ln[U'(W_T)] = -\alpha W_T,$$

which is normal since W_T is normal. Also, given that W_T is normal,

$$E_t[U^{'}(W_T)] = exp\left\{-\alpha\mu_w + \frac{\alpha^2}{2}\sigma_w^2\right\} \tag{10}$$

Since W_T and X_T are joint normal we can write the linear regression:

$$-\alpha W_T = a + b\ X_T + \epsilon \tag{11}$$

where X_T is independent of ϵ. From (11) it follows, taking variances

$$\alpha^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \alpha^2\sigma_w^2 - b^2\sigma_x^2 \tag{12}$$

Also, from (11) it follows, taking expectations that

$$-\alpha\mu_w = a + b\mu_x$$

and hence the conditional expectation

$$E_t[\ln(U^{'}(W_T))|X_T] = a + b\, X_T = -\alpha\mu_w - b\mu_x + b\, X_T. \tag{13}$$

Using equations (12) and and (13), it follows that

$$\begin{aligned}
\psi_{t,T} &= E_t\left[U^{'}(W_T) \mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[-\alpha\mu_w - b\mu_x + b\, X_T + (\alpha^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[-\alpha\mu_w + (\alpha^2\sigma_w^2)/2\right]} \\
&= Ae^{bX_T}.
\end{aligned}$$

Hence, the asset specific pricing kernel is an exponential function of the cash flow X_T and the price of X_T therefore follows an arithmetic random walk. $\square$

2. Cases 2 and 3

$$g(W_T) = \ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right]$$

with $\theta = 0$, in case 2, implies by Lemma 1 that

$$U'(W_T) = \left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}$$

is lognormal, and has logarithmic mean

$$E_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi\mu_w - \varphi\,\ln\left[(1-\varphi)\alpha^{-1}\right]$$

and logarithmic variance

$$Var_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi^2\sigma_w^2$$

It follows that

$$E_t\left[U^{'}(W_T)\right] = exp\left\{\varphi\mu_w - \varphi\,\ln\left[(1-\varphi)\alpha^{-1}\right] + \frac{1}{2}\varphi^2\sigma_w^2\right\}. \tag{14}$$

Since W_T and X_T are joint displaced lognormal-normal, $\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)$ and X_T are joint normal, and hence we can write the linear regression:

$$\varphi\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right) = a + b\, X_T + \epsilon \tag{15}$$

where X_T is independent of ϵ. From (15) it follows, taking variances

$$\varphi^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \varphi^2\sigma_w^2 - b^2\sigma_x^2. \tag{16}$$

Also, from (15) it follows, taking expectations that

$$\varphi\mu_w - \varphi \ln\left[(1-\varphi)\alpha^{-1}\right] = a + b\mu_x$$

and hence the conditional expectation

$$E_t[\ln(U^{'}(W_T))|X_T] = a + b\ X_T = \varphi\mu_w - \varphi \ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ X_T. \tag{17}$$

Using equations (16) and and (17), it follows that

$$\begin{aligned}
\psi_{t,T} &= E_t\left[U^{'}(W_T) \mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + bX_T + (\varphi^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] + (\varphi^2\sigma_w^2)/2\right]} \\
&= Ae^{bX_T}.
\end{aligned}$$

Hence, the asset specific pricing kernel is an exponential function of the cash flow X_T and the price of X_T therefore follows an arithmetic random walk. In particular, when in case 2 $\theta = 0$ and wealth W_T and X_T are joint lognormal-normal, the price of X_T again follows an arithmetic random walk.

□

The implication of Theorem 2 is that if any of the three conditions hold, then a risk-neutral-valuation relationship holds for the valuation of options on assets, where the conditional expectation of the price of the asset at time T follows an arithmetic, normally distributed random walk. Hence the condiitions for the Brennan (1979) model to hold for options on normally distributed asset prices are somewhat wider than those found by Brennan.

We now extend the analysis to assets which follow displaced-geometric random walks (DGRW) as in Rubinstein (1983) and negatively-skewed-geometric random walks (NSGRW) as in Stapleton and Subrahmanyam (1993). We state the two cases as one theorem. We have:

Theorem 3 *[DGRW (NSGRW) for the forward price]*

Assume that a representative agent exists with a utility function of the HARA family. Let the conditional expectation at time t of the asset price X_T follow a displaced-geometric (negatively-skewed-geometric) random walk process, so that:

$$\begin{aligned} \ln[E_t(X_T - \beta)] &\sim N[\mu_x, \sigma_x], \qquad t \leq T \\ (\ln[\beta - E_t(X_T] &\sim N[\mu_x, \sigma_x], \qquad t \leq T) \end{aligned}$$

Then the forward price $F_t(X_T)$ follows a DGRW (NSGRW) if either:

1. $\varphi \to -\infty$, $\theta = 1$, *and* $\alpha > 0$; *that is preferences are characterised by a negative exponential utility function, and wealth W_T and X_T are joint normal-displaced (negatively-skewed) lognormal.*

2. $\alpha > 0, \varepsilon < 0, \theta = 0$, *that is preferences are CPRA, and wealth W_T and X_T are joint lognormal-displaced (negatively-skewed) lognormal.*

3. $\alpha > 0$, $\varphi < 0$, *and* $W_T > -\frac{(1-\varphi)\theta}{\alpha}$; *that is preferences are characterised by an extended power utility function, and wealth W_T and X_T are joint displaced lognormal-displaced (negatively skewed) lognormal.*

Proof

From Lemma 2 (3) a sufficient condition is that the asset-specific pricing kernel has the form $\psi_{t,T} = a(X_T - \beta)^\alpha$ ($\psi_{t,T} = a(\beta - X_T)^\alpha$). Using a similar argument to that used in the proof of Theorem 2 it is then straightforward to show that if either of conditions 1, 2, or 3 obtain then the forward price of X_T follows a DGRW (NSGRW). The details are shown in the appendix.□

7 Conclusions

We have shown that the same conditions lead to the preservation of random walks and risk-neutral-valuation relationships for option pricing in the case of assets with lognormal, normal, displaced lognormal and negatively-skewed lognormal distributions. In an economy, different assets may well follow different processes. However, the key to whether forward prices also follow these processes lies with the distribution of aggregate wealth and the utility function of the representative investor. We have shown that risk-neutral-valuation relationships hold for each asset class if either one of three conditions hold. Either wealth is normally distributed and utility is of the constant absolute risk averse type. Or, wealth is lognormally distributed and wealth is of the constant proportional risk averse type. Alternatively, wealth may be displaced lognormal and wealth is of the HARA class, with a particular form.

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8 Appendix: Proof of Theorem 3

The proof is similar to the proof of Theorem 1. From Lemma 2 (3) a sufficient condition is that the asset-specific pricing kernel has the form $\psi_{t,T} = A(X_T - \beta)^\alpha$ ($\psi_{t,T} = A(\beta - X_T)^\alpha$), for constants A and b.

We will show that any of the conditions 1), 2), and 3) above are sufficient for this condition to hold.

1. $g(W_T) = W_T$ means that $W_T \sim N(\mu_w, \sigma_w^2)$. $\varphi \to -\infty$, $\theta = 1$, $\alpha > 0$ implies from Lemma 1 that

$$U'(W_T) = e^{-\alpha W_T}.$$

Hence

$$\ln[U'(W_T)] = -\alpha W_T,$$

which is normal since W_T is normal. Also, since W_T is normal,

$$E_t[U^{'}(W_T)] = exp\left\{-\alpha\mu_w + \frac{\alpha^2}{2}\sigma_w^2\right\} \tag{18}$$

Since W_T and $\ln(X_T - \beta)$ ($\ln(\beta - X_T)$) are joint normal we can write the linear regression:

$$\begin{aligned} -\alpha W_T &= a + b\ln(X_T - \beta) + \epsilon \qquad (19)\\ (-\alpha W_T &= a + b\ln(\beta - X_T) + \epsilon) \end{aligned}$$

where $\ln(X_T - \beta)(\ln(\beta - X_T))$ is independent of ϵ. From (19) it follows, taking variances

$$\alpha^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \alpha^2\sigma_w^2 - b^2\sigma_x^2 \tag{20}$$

Also, from (19) it follows, taking expectations that

$$-\alpha\mu_w = a + b\mu_x$$

and hence the conditional expectation

$$
\begin{aligned}
E_t[\ln(U^{'}(W_T))|X_T] &= a + b\ln(X_T-\beta) = -\alpha\mu_w - b\mu_x + b\ln(X_T-\beta) \\
(E_t[\ln(U^{'}(W_T))|X_T] &= a + b\ln(\beta-X_T) = -\alpha\mu_w - b\mu_x + b\ln(\beta-X_T)).
\end{aligned}
$$

Substituting, it then follows that

$$
\begin{aligned}
\psi_{t,T} &= E_t\left[U^{'}(W_T)\mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[-\alpha\mu_w - b\mu_x + b\ln(X_T-\beta) + (\alpha^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[-\alpha\mu_w + (\alpha^2\sigma_w^2)/2\right]} \\
&= A(\beta - X_T)^b. \\
(\psi_{t,T} &= E_t\left[U^{'}(W_T)\mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[-\alpha\mu_w - b\mu_x + b\ln(\beta-X_T) + (\alpha^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[-\alpha\mu_w + (\alpha^2\sigma_w^2)/2\right]} \\
&= A(X_T - \beta)^b.)
\end{aligned}
$$

□

2. Cases 2 and 3

$$
g(W_T) = \ln\left[W_T + \frac{(1-\varphi)\theta}{\alpha}\right]
$$

with $\theta = 0$, in case 2, implies by Lemma 1 that

$$
U'(W_T) = \left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}
$$

is lognormal, and has logarithmic mean

$$
E_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right]
$$

and logarithmic variance

$$
Var_t\left[\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)^{\varphi}\right] = \varphi^2\sigma_w^2
$$

It follows that

$$
E_t\left[U^{'}(W_T)\right] = exp\left\{\varphi\mu_w - \varphi\ln\left[(1-\varphi)\alpha^{-1}\right] + \frac{1}{2}\varphi^2\sigma_w^2\right\}. \tag{21}
$$

Since W_T and X_T are joint displaced lognormal-displaced lonormal (negatively skewed lognormal), $\ln\left(\frac{\alpha W_T}{1-\varphi} + \theta\right)$ and $\ln(X_T-\beta)(\ln(\beta-X_T))$ are joint normal, and hence we can write the linear regression:

$$
\begin{aligned}
\varphi \ln\left(\frac{\alpha W_T}{1-\varphi}+\theta\right) &= a + b\ \ln(X_T-\beta)+\epsilon \\
(\varphi \ln\left(\frac{\alpha W_T}{1-\varphi}+\theta\right) &= a + b\ \ln(\beta - X_T)+\epsilon)
\end{aligned}
$$

where X_T is independent of ϵ. It then follows, taking variances

$$\varphi^2\sigma_w^2 = b^2\sigma_x^2 + \sigma_\epsilon^2$$

and hence

$$var_t[\ln(U^{'}(W_T))|X_T] = \sigma_\epsilon^2 = \varphi^2\sigma_w^2 - b^2\sigma_x^2. \tag{22}$$

Also, from (22) it follows, taking expectations that

$$\varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] = a + b\mu_x$$

and hence the conditional expectation

$$
\begin{aligned}
E_t[\ln(U^{'}(W_T))|X_T] &= a + b\ \ln(X_T-\beta) = \varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ \ln(X_T-\beta). \\
(E_t[\ln(U^{'}(W_T))|X_T] &= a + b\ \ln(\beta-X_T) = \varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ \ln(\beta-X_T).)
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\psi_{t,T} &= E_t\left[U^{'}(W_T)\mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[\varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ln(X_T-\beta) + (\varphi^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[\varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] + (\varphi^2\sigma_w^2)/2\right]} \\
&= A(X_T-\beta)^b. \\
(\psi_{t,T} &= E_t\left[U^{'}(W_T)\mid X_T\right]/E_t\left[U^{'}(W_T)\right] \\
&= \frac{exp\left[\varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] - b\mu_x + b\ln(\beta-X_T) + (\varphi^2\sigma_w^2 - b^2\sigma_x^2)/2\right]}{exp\left[\varphi\mu_w - \varphi\ \ln\left[(1-\varphi)\alpha^{-1}\right] + (\varphi^2\sigma_w^2)/2\right]} \\
&= A(\beta-X_T)^b).
\end{aligned}
$$

□