

The Pricing and Hedging of Interest-Rate Derivatives: Theory and Practice

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April 28, 2005

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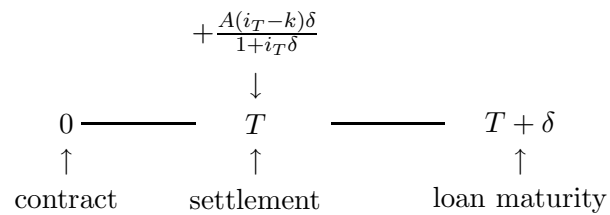
1 Definition of Interest-Rate Derivative Contracts

In this section we provide definitions (using cash flow diagrams) of the basic vanilla interest-rate derivatives. These can be classified as single period (FRA, caplet, Libor futures, option on Libor futures) and as multi-period (swaps, caps, swaptions).

Forward Rate Agreement (FRA)

A forward rate agreement (FRA) is an agreement to exchange fixed-rate interest payments at a rate k for Libor payments, on a principal amount A for the loan period T to $T + \delta$.

The time scale for payments on a T -maturity FRA is shown below:



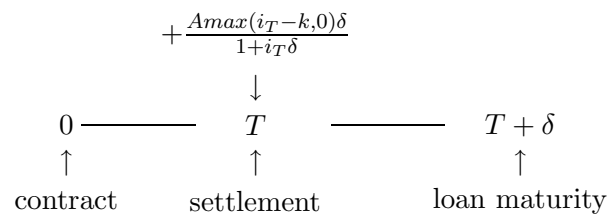
Here, $t = 0$ is the contract agreement date, T is the settlement date for the contract, and $T + \delta$ is the date on which the notional loan underlying the FRA is repaid.

- A is the principal of the underlying loan
- i_t is the spot Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Interest-Rate Caplet

An interest-rate caplet (floorlet) is an option to enter a long (short) FRA at time T at a fixed rate k

The time scale for payments on a T -maturity caplet is shown below:



Here, $t = 0$ is the contract agreement date, T is the settlement date, and $T + \delta$ is the date on which the notional loan underlying the FRA is repaid.

- A is the principal of the underlying loan
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Interest-Rate Futures

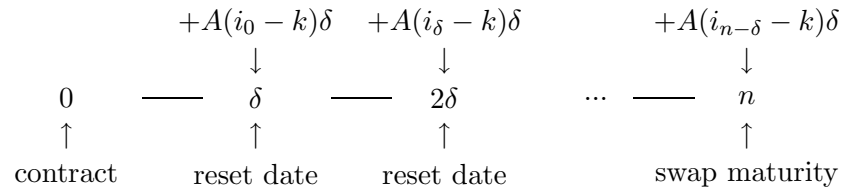
A Libor Futures contract is an agreement made at time $t = 0$ to pay or receive the difference between $H_{t,T}$, the futures price at time t and the price at time $t + 1$, $H_{t+1,T}$, daily until the maturity of the contract. The daily cash flows on a T -maturity long futures, contracted at $t = 0$ at a futures price $H_{0,T}$ is shown below:

$$\begin{array}{ccccccccccc}
 +(H_{1,T} - H_{0,T})A\delta & \dots & +(H_{t+1,T} - H_{t,T})A\delta & \dots & & & & & & & +(H_{T,T} - H_{T-1,T})A\delta \\
 \downarrow & & & & \downarrow & & & & & & \downarrow \\
 0 & 1 & \dots & \dots & t & t+1 & \dots & \text{---} & & & T \\
 \uparrow & & & & & \uparrow & & & & & \uparrow \\
 \text{contract} & & & & \text{mark-to-market dates} & & & & & & \text{futures maturity}
 \end{array}$$

- $H_{t,T}$ is the Libor futures price at time t
- A is the principal of the contract
- δ is the loan period

Interest-Rate Swap

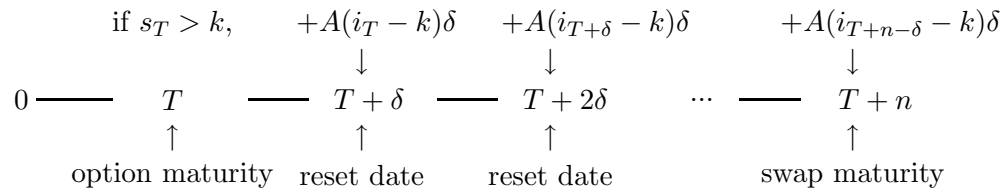
An interest-rate swap is an agreement made at time 0 to exchange fixed-rate interest payments at a rate k for Libor payments, on a principal amount A every δ years, over the loan period 0 to n .



- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Swaption

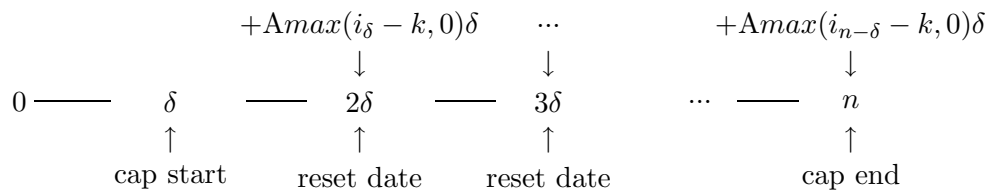
A European-style swaption, with strike rate k , gives the right to enter an n -year swap on the option maturity date T . The cash flows on a pay-fixed swaption (payer swaption) are shown below.



- s_T is the swap rate at time T
- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Interest-Rate Cap

An interest-rate cap, with strike rate k , gives the right to enter a series of FRAs every δ years over n years.



- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

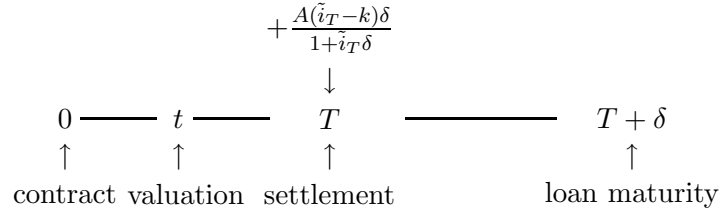
An interest-rate cap is equivalent to a portfolio of put options on one-period zero-coupon bonds.

2 The Valuation of Interest-Rate Derivative Contracts

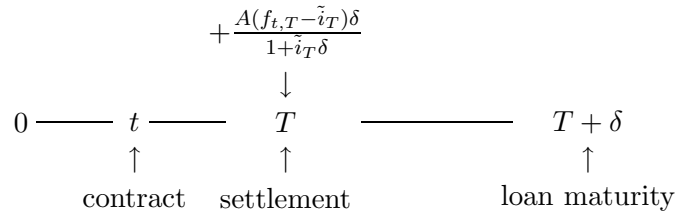
The definitions in the previous section show the cash flows given that the derivative is contracted at date $t = 0$. In this section we value these products at a date t , which as a special case could be $t = 0$. The FRAs and swaps are valued using standard discounting techniques. The options are valued using different versions of the Black model.

Valuation of an FRA

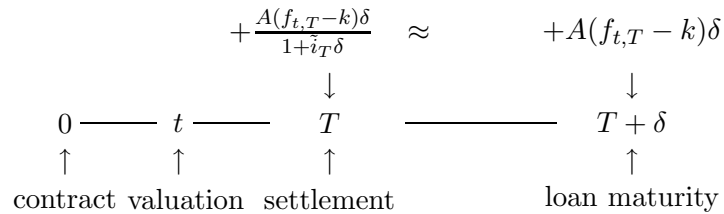
1. Long (time $t = 0$ contract) FRA Payoff



2. Short (time t contract) FRA Payoff



3. Reversed FRA Payoff



FRA value at time t :

$$FRA_{t,T,\delta}(k) = A(f_{t,T} - k)\delta B_{t,T+\delta}$$

- $f_{t,T}$ is the Libor forward rate at time t for delivery at T
- $B_{t,T+\delta}$ is the price at t of a bond paying \$1 at $T + \delta$

The Black Model for a Caplet

If the BGM process for forward rates holds, the value of a caplet at time t , with maturity T is given by:

$$\text{caplet}_{t,T,\delta}(k) = \frac{A}{1 + f_{t,T}\delta} \delta [f_{t,T}N(d_1) - kN(d_2)] B_{t,T} \quad (1)$$

where

$$d_1 = \frac{\ln\left(\frac{f_{t,T}}{k}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\tau = T - t$$

- $\text{caplet}_{t,T,\delta}(k)$ is the value at t of a T -year caplet at a strike rate k
- $f_{t,T}$ is the Libor forward rate at time t for delivery at T
- $B_{t,T}$ is the price at t of a bond paying \$1 at T
- σ is the volatility of the Libor, i_T

The Black Model for Libor Futures Options

Assuming these are European-style, marked-to-market options, then a put on the futures price has a futures value

$$P_{t,T}(k) = [(1 - H_{t,T})N(d_1) - (1 - K)N(d_2)]$$

where

$$d_1 = \frac{\ln\left(\frac{1-H_{t,T}}{1-K}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where $H_{t,T}$ is the futures price and K is the strike price.

The futures price of the option can be established for Libor options by assuming that the futures *rate* follows a lognormal diffusion process (limit of the geometric binomial process as $n \rightarrow \infty$).

- $P_{t,T}(k)$ is the value at t of a T -year futures option at a strike rate k
- $H_{t,T}$ is the Libor futures price at time t for delivery at T
- σ is the volatility of Libor

The Black Model for Caplets/Floorlets using Equivalent Bond Options

If the (continuously compounded) interest rate is normally distributed, the price of a floorlet on Libor is given by

$$\text{Floorlet}_{t,T,\delta}(k) = [B_{t,T+\delta}N(d_1) - KB_{t,T}N(d_2)](1 + k\delta)$$

where

$$d_1 = \frac{\ln\left(\frac{F_{t,T,T+\delta}}{K}\right) + \frac{\sigma'^2\tau}{2}}{\sigma'\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma'\sqrt{\tau}$$

$$K = \frac{1}{1 + k\delta}$$

$$\tau = T - t$$

Here, the bond forward price is assumed to follow a lognormal diffusion process.

- $\text{floorlet}_{t,T,\delta}(k)$ is the value at t of a T -year floorlet at a strike rate k
- $B_{t,T+i\delta}$ is the price at t of a bond paying \$1 at $T + i\delta$
- σ' is the volatility of the zero-coupon bond price
- K is the strike price of the equivalent bond option

3 Parity Relationships for Interest-Rate Options

Notation

- P_t value of put option at time t
- C_t value of call option at time t
- K strike price
- T option maturity

A long FRA at k is equivalent to $(1 + k\delta)$ short forward contracts on a one-period zero-coupon bond.

A caplet on Libor at k is equivalent to $(1 + k\delta)$ put options on a one-period zero-coupon bond.

An long interest-rate swap (to pay fixed, receive Libor) is equivalent to a short forward contract on an n -year coupon bond, with coupon k .

A European-style payer swaption (to pay fixed, receive Libor) is equivalent to a put option on an n -year coupon bond, with coupon k , and with a strike price of par.