

# The Pricing and Hedging of Interest-Rate Derivatives: Theory and Practice

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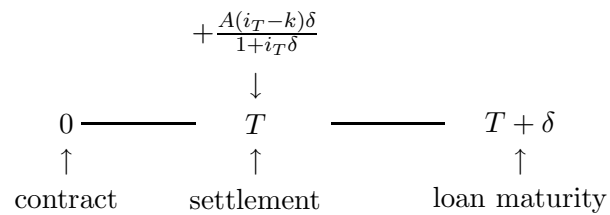
# **1 Definition of Interest-Rate Derivative Contracts**

In this section we provide definitions (using cash flow diagrams) of the basic vanilla interest-rate derivatives. These can be classified as single period (FRA, caplet, Libor futures, option on Libor futures) and as multi-period (swaps, caps, swaptions).

### Forward Rate Agreement (FRA)

A forward rate agreement (FRA) is an agreement to exchange fixed-rate interest payments at a rate  $k$  for Libor payments, on a principal amount  $A$  for the loan period  $T$  to  $T + \delta$ .

The time scale for payments on a  $T$ -maturity FRA is shown below:



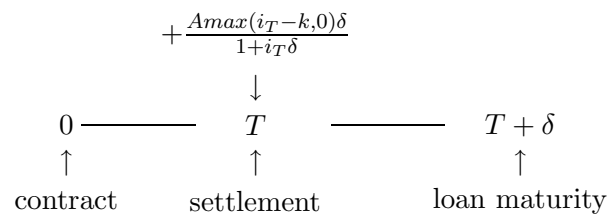
Here,  $t = 0$  is the contract agreement date,  $T$  is the settlement date for the contract, and  $T + \delta$  is the date on which the notional loan underlying the FRA is repaid.

- $A$  is the principal of the underlying loan
- $i_t$  is the spot Libor interest rate at time  $t$
- $k$  is the strike rate of the contract
- $\delta$  is the loan period

### Interest-Rate Caplet

An interest-rate caplet (floorlet) is an option to enter a long (short) FRA at time  $T$  at a fixed rate  $k$

The time scale for payments on a  $T$ -maturity caplet is shown below:



Here,  $t = 0$  is the contract agreement date,  $T$  is the settlement date, and  $T + \delta$  is the date on which the notional loan underlying the FRA is repaid.

- $A$  is the principal of the underlying loan
- $i_t$  is the spot Libor interest rate at time  $t$
- $k$  is the strike rate of the contract
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**Interest-Rate Futures**

A Libor Futures contract is an agreement made at time  $t = 0$  to pay or receive the difference between  $H_{t,T}$ , the futures price at time  $t$  and the price at time  $t + 1$ ,  $H_{t+1,T}$ , daily until the maturity of the contract. The daily cash flows on a  $T$ -maturity long futures, contracted at  $t = 0$  at a futures price  $H_{0,T}$  is shown below:

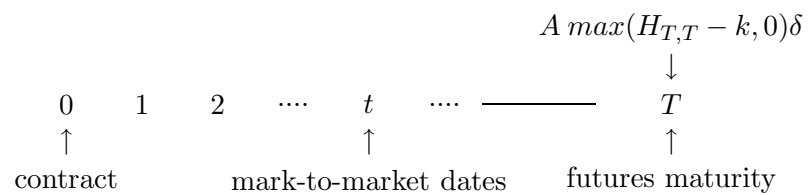
$$\begin{array}{ccccccccccc}
 +(H_{1,T} - H_{0,T})A\delta & \dots & + (H_{t+1,T} - H_{t,T})A\delta & \dots & \dots & \dots & + (H_{T,T} - H_{T-1,T})A\delta \\
 \downarrow & & & & \downarrow & & & & \downarrow \\
 0 & 1 & \dots & \dots & t & t+1 & \dots & \text{---} & T \\
 \uparrow & & & & & \uparrow & & & \uparrow \\
 \text{contract} & & & & \text{mark-to-market dates} & & & & \text{futures maturity}
 \end{array}$$

- $H_{t,T}$  is the Libor futures price at time  $t$
- $A$  is the principal of the contract
- $\delta$  is the loan period

### Options on Interest-Rate Futures

A *Libor Futures option* is an option to enter an interest-rate futures contract at a fixed rate  $k$

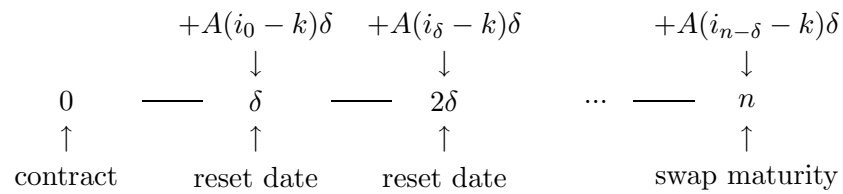
A European-style futures call option with maturity  $T$  has a payoff:



- $H_{T,T}$  is the Libor futures price at time  $T$
- $A$  is the principal of the contract
- $\delta$  is the loan period
- $k$  is the strike rate

### Interest-Rate Swap

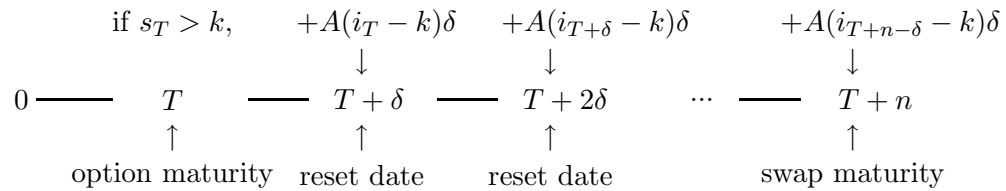
An interest-rate swap is an agreement made at time 0 to exchange fixed-rate interest payments at a rate  $k$  for Libor payments, on a principal amount  $A$  every  $\delta$  years, over the loan period 0 to  $n$ .



- $A$  is the principal of the underlying loan
- $i_t$  is the Libor interest rate at time  $t$
- $k$  is the strike rate of the contract
- $\delta$  is the loan period

### Swaption

A European-style swaption, with strike rate  $k$ , gives the right to enter an  $n$ -year swap on the option maturity date  $T$ . The cash flows on a pay-fixed swaption (payer swaption) are shown below.

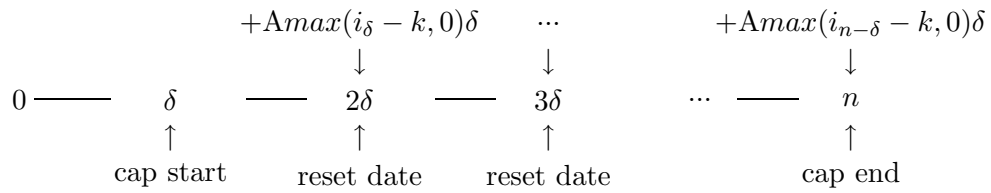


- $s_T$  is the swap rate at time  $T$
- $A$  is the principal of the underlying loan
- $i_t$  is the Libor interest rate at time  $t$
- $k$  is the strike rate of the contract
- $\delta$  is the loan period



### Interest-Rate Cap

An interest-rate cap, with strike rate  $k$ , gives the right to enter a series of FRAs every  $\delta$  years over  $n$  years.



- $A$  is the principal of the underlying loan
- $i_t$  is the Libor interest rate at time  $t$
- $k$  is the strike rate of the contract
- $\delta$  is the loan period

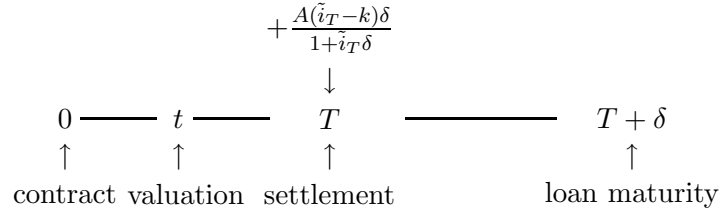
An interest-rate cap is equivalent to a portfolio of put options on one-period zero-coupon bonds.

## **2 The Valuation of Interest-Rate Derivative Contracts**

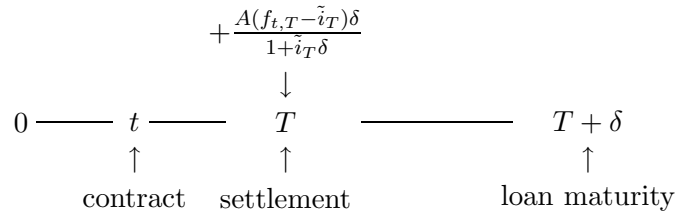
The definitions in the previous section show the cash flows given that the derivative is contracted at date  $t = 0$ . In this section we value these products at a date  $t$ , which as a special case could be  $t = 0$ . The FRAs and swaps are valued using standard discounting techniques. The options are valued using different versions of the Black model.

**Valuation of an FRA**

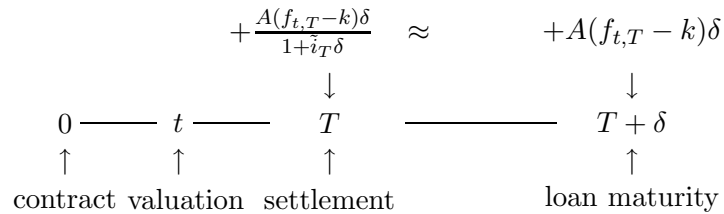
1. Long (time  $t = 0$  contract) FRA Payoff



2. Short (time  $t$  contract) FRA Payoff



3. Reversed FRA Payoff



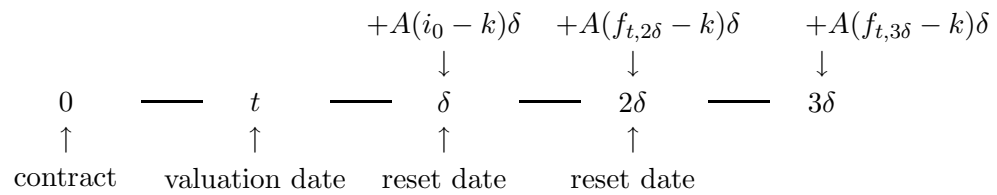
FRA value at time  $t$ :

$$FRA_{t,T,\delta}(k) = A(f_{t,T} - k)\delta B_{t,T+\delta}$$

- $f_{t,T}$  is the Libor forward rate at time  $t$  for delivery at  $T$
- $B_{t,T+\delta}$  is the price at  $t$  of a bond paying \$1 at  $T + \delta$

**Valuation of an Interest-Rate Swap**

Assuming that the valuation date  $t$  is  $0 < t < \delta$ :

**Swap Reversed Cash Flows****Swap: Value at Time  $t$** 

$$\begin{aligned} swap_{t,0,n,\delta}(k) = & \\ & A(i_0 - k)\delta B_{t,\delta} \\ & + A(f_{t,2\delta} - k)\delta B_{t,2\delta} \\ & + A(f_{t,3\delta} - k)\delta B_{t,3\delta} \end{aligned}$$

A horizontal timeline diagram showing the valuation date  $t$  and subsequent reset dates  $\delta$ ,  $2\delta$ , and  $3\delta$ . The timeline is represented by a series of horizontal lines (represented by dashes) connecting the points 0,  $t$ ,  $\delta$ ,  $2\delta$ , and  $3\delta$  on the top axis. Below these points are labels: "contract" under 0, "valuation date" under  $t$ , "reset date" under  $\delta$ , "reset date" under  $2\delta$ , and "reset date" under  $3\delta$ . Upward-pointing arrows are located at each of these points. A downward-pointing arrow is shown between  $t$  and  $\delta$ , pointing to the equation above.

- $swap_{t,0,n,\delta}(k)$  is the value at  $t$  of an  $n$ -year  $\delta$ -reset swap contracted at time  $t = 0$  at a strike rate  $k$
- $f_{t,T}$  is the Libor forward rate at time  $t$  for delivery at  $T$
- $B_{t,i\delta}$  is the price at  $t$  of a bond paying \$1 at  $i\delta$

### The Black Model for a Caplet

If the BGM process for forward rates holds, the value of a caplet at time  $t$ , with maturity  $T$  is given by:

$$caplet_{t,T,\delta}(k) = \frac{A}{1 + f_{t,T}\delta} \delta [f_{t,T}N(d_1) - kN(d_2)] B_{t,T} \quad (1)$$

where

$$d_1 = \frac{\ln\left(\frac{f_{t,T}}{k}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\tau = T - t$$

- $caplet_{t,T,\delta}(k)$  is the value at  $t$  of a  $T$ -year caplet at a strike rate  $k$
- $f_{t,T}$  is the Libor forward rate at time  $t$  for delivery at  $T$
- $B_{t,T}$  is the price at  $t$  of a bond paying \$1 at  $T$
- $\sigma$  is the volatility of the Libor,  $i_T$

**The Black Model for Libor Futures Options**

Assuming these are European-style, marked-to-market options, then a put on the futures price has a futures value

$$P_{t,T}(k) = [(1 - H_{t,T})N(d_1) - (1 - K)N(d_2)]$$

where

$$d_1 = \frac{\ln\left(\frac{1-H_{t,T}}{1-K}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where  $H_{t,T}$  is the futures price and  $K$  is the strike price.

The futures price of the option can be established for Libor options by assuming that the futures *rate* follows a lognormal diffusion process (limit of the geometric binomial process as  $n \rightarrow \infty$ ).

- $P_{t,T}(k)$  is the value at  $t$  of a  $T$ -year futures option at a strike rate  $k$
- $H_{t,T}$  is the Libor futures price at time  $t$  for delivery at  $T$
- $\sigma$  is the volatility of Libor

### The Black Model for Caplets/Floorlets using Equivalent Bond Options

If the (continuously compounded) interest rate is normally distributed, the price of a floorlet on Libor is given by

$$\text{Floorlet}_{t,T,\delta}(k) = [B_{t,T+\delta}N(d_1) - KB_{t,T}N(d_2)](1 + k\delta)$$

where

$$d_1 = \frac{\ln\left(\frac{F_{t,T,T+\delta}}{K}\right) + \frac{\sigma'^2\tau}{2}}{\sigma'\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma'\sqrt{\tau}$$

$$K = \frac{1}{1 + k\delta}$$

$$\tau = T - t$$

Here, the bond forward price is assumed to follow a lognormal diffusion process.

- $\text{floorlet}_{t,T,\delta}(k)$  is the value at  $t$  of a  $T$ -year floorlet at a strike rate  $k$
- $B_{t,T+i\delta}$  is the price at  $t$  of a bond paying \$1 at  $T + i\delta$
- $\sigma'$  is the volatility of the zero-coupon bond price
- $K$  is the strike price of the equivalent bond option

### 3 Parity Relationships for Interest-Rate Options

Notation

- $P_t$  value of put option at time  $t$
- $C_t$  value of call option at time  $t$
- $K$  strike price
- $T$  option maturity

A long FRA at  $k$  is equivalent to  $(1 + k\delta)$  short forward contracts on a one-period zero-coupon bond.

A caplet on Libor at  $k$  is equivalent to  $(1 + k\delta)$  put options on a one-period zero-coupon bond.

An long interest-rate swap (to pay fixed, receive Libor) is equivalent to a short forward contract on an  $n$ -year coupon bond, with coupon  $k$ .

A European-style payer swaption (to pay fixed, receive Libor) is equivalent to a put option on an  $n$ -year coupon bond, with coupon  $k$ , and with a strike price of par.