The Pricing and Hedging of Interest-Rate Derivatives: Theory and Practice

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1 Definition of Interest-Rate Derivative Contracts

In this section we provide definitions (using cash flow diagrams) of the basic vanilla interest-rate derivatives. These can be classified as single period (FRA, caplet, Libor futures, option on Libor futures) and as multi-period (swaps, caps, swaptions).

Forward Rate Agreement (FRA)

A forward rate agreement (FRA) is an agreement to exchange fixed-rate interest payments at a rate k for Libor payments, on a principal amount A for the loan period T to $T+\delta$.

The time scale for payments on a T-maturity FRA is shown below:

$$\begin{array}{ccccc} & & +\frac{A(i_T-k)\delta}{1+i_T\delta} & & \\ & & \downarrow & & \\ 0 & \text{———} & T & \text{———} & T+\delta \\ \uparrow & & \uparrow & & \uparrow \\ \text{contract} & & \text{settlement} & & \text{loan maturity} \end{array}$$

Here, $t=0$ is the contract agreement date, T is the settlement date for the contract, and $T+\delta$ is the date on which the notional loan underlying the FRA is repaid.

- A is the principal of the underlying loan
- i_t is the spot Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Interest-Rate Caplet

An interest-rate caplet (floorlet) is an option to enter a long (short) FRA at time T at a fixed rate k

The time scale for payments on a T-maturity caplet is shown below:

$$\begin{array}{ccccc} & & +\frac{Amax(i_T-k,0)\delta}{1+i_T\delta} & & \\ & & \downarrow & & \\ 0 & \text{———} & T & \text{———} & T+\delta \\ \uparrow & & \uparrow & & \uparrow \\ \text{contract} & & \text{settlement} & & \text{loan maturity} \end{array}$$

Here, $t = 0$ is the contract agreement date, T is the settlement date, and $T + \delta$ is the date on which the notional loan underlying the FRA is repaid.

- A is the principal of the underlying loan
- i_t is the spot Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Interest-Rate Futures

A Libor Futures contract is an agreement made at time $t = 0$ to pay or receive the difference between $H_{t,T}$, the futures price at time t and the price at time $t+1$, $H_{t+1,T}$, daily until the maturity of the contract. The daily cash flows on a T-maturity long futures, contracted at $t = 0$ at a futures price $H_{0,T}$ is shown below:

$$
\begin{array}{cccccccc}
\multicolumn{3}{c}{+(H_{1,T} - H_{0,T})A\delta..} & \multicolumn{3}{c}{+(H_{t+1,T} - H_{t,T})A\delta....} & \multicolumn{2}{c}{+(H_{T,T} - H_{T-1,T})A\delta} \\
 & \downarrow & & & \downarrow & & & \downarrow \\
0 & 1 & \cdots & \cdots\ t & t+1 & \cdots & \text{———} & T \\
\uparrow & & & & \uparrow & & & \uparrow \\
\text{contract} & & & & \text{mark-to-market dates} & & & \text{futures maturity}
\end{array}
$$

- $H_{t,T}$ is the Libor futures price at time t
- A is the principal of the contract
- δ is the loan period

Options on Interest-Rate Futures

A Libor Futures option is an option to enter an interest-rate futures contract at a fixed rate k

A European-style futures call option with maturity T has a payoff:

$$
\begin{array}{cccccccc}
 & & & & & & & A\,max(H_{T,T}-k,0)\delta \\
 & & & & & & & \downarrow \\
0 & 1 & 2 & \cdots & t & \cdots & \text{———} & T \\
\uparrow & & & & \uparrow & & & \uparrow \\
\text{contract} & & & & \text{mark-to-market dates} & & & \text{futures maturity}
\end{array}
$$

- $H_{T,T}$ is the Libor futures price at time T
- A is the principal of the contract
- δ is the loan period
- k is the strike rate

Interest-Rate Swap

An interest-rate swap is an agreement made at time 0 *to exchange fixed-rate interest payments at a rate* k *for Libor payments, on a principal amount* A *every* δ *years, over the loan period* 0 *to* n.

$$\begin{array}{ccccccccc}
 & & +A(i_0-k)\delta & & +A(i_\delta-k)\delta & & & & +A(i_{n-\delta}-k)\delta \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \text{———} & \delta & \text{———} & 2\delta & & \cdots & \text{———} & n \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\text{contract} & & \text{reset date} & & \text{reset date} & & & & \text{swap maturity}
\end{array}$$

- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Swaption

A European-style swaption, with strike rate k, gives the right to enter an n-year swap on the option maturity date T. The cash flows on a pay-fixed swaption (payer swaption) are shown below.

$$
\begin{array}{cccccccccc}
 & \text{if } s_T > k, & & +A(i_T - k)\delta & & +A(i_{T+\delta} - k)\delta & & & & +A(i_{T+n-\delta} - k)\delta \\
 & & & \downarrow & & \downarrow & & & & \downarrow \\
0 \text{ ——} & T & \text{——} & T+\delta & \text{——} & T+2\delta & & \cdots & \text{——} & T+n \\
 & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
 & \text{option maturity} & & \text{reset date} & & \text{reset date} & & & & \text{swap maturity}
\end{array}
$$

- s_T is the swap rate at time T
- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

Interest-Rate Cap

An interest-rate cap, with strike rate k, gives the right to enter a series of FRAs every δ years over n years.

$$\begin{array}{cccccccc}
 & & & +\mathrm{A}max(i_{\delta} - k, 0)\delta & & \cdots & & +\mathrm{A}max(i_{n-\delta} - k, 0)\delta \\
 & & & \downarrow & & \downarrow & & \downarrow \\
0 \text{ ——} & \delta & \text{——} & 2\delta & \text{——} & 3\delta & \cdots \text{ ——} & n \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & \text{cap start} & & \text{reset date} & & \text{reset date} & & \text{cap end}
\end{array}$$

- A is the principal of the underlying loan
- i_t is the Libor interest rate at time t
- k is the strike rate of the contract
- δ is the loan period

An interest-rate cap is equivalent to a portfolio of put options on one-period zero-coupon bonds.

2 The Valuation of Interest-Rate Derivative Contracts

The definitions in the previous section show the cash flows given that the derivative is contracted at date $t = 0$. In this section we value these products at a date t, which as a special case could be $t = 0$. The FRAs and swaps are valued using standard discounting techniques. The options are valued using different versions of the Black model.

Valuation of an FRA

1. Long (time $t = 0$ contract) FRA Payoff

$$\begin{array}{ccccccc}
 & & & & +\frac{A(\tilde{i}_T - k)\delta}{1+\tilde{i}_T\delta} & & \\
 & & & & \downarrow & & \\
0 & \text{———} & t & \text{———} & T & \text{————} & T+\delta \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{contract} & & \text{valuation} & & \text{settlement} & & \text{loan maturity}
\end{array}$$

2. Short (time t contract) FRA Payoff

$$\begin{array}{ccccccc}
 & & & & +\frac{A(f_{t,T} - \tilde{i}_T)\delta}{1+\tilde{i}_T\delta} & & \\
 & & & & \downarrow & & \\
0 & \text{———} & t & \text{———} & T & \text{————} & T+\delta \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \text{contract} & & \text{settlement} & & \text{loan maturity}
\end{array}$$

3. Reversed FRA Payoff

$$\begin{array}{ccccccc}
 & & & & +\frac{A(f_{t,T} - k)\delta}{1+\tilde{i}_T\delta} & \approx & +A(f_{t,T} - k)\delta \\
 & & & & \downarrow & & \downarrow \\
0 & \text{———} & t & \text{———} & T & \text{————} & T+\delta \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{contract} & & \text{valuation} & & \text{settlement} & & \text{loan maturity}
\end{array}$$

FRA value at time t:

$$FRA_{t,T,\delta}(k) = A(f_{t,T} - k)\delta B_{t,T+\delta}$$

- $f_{t,T}$ is the Libor forward rate at time t for delivery at T
- $B_{t,T+\delta}$ is the price at t of a bond paying \$1 at $T + \delta$

Valuation of an Interest-Rate Swap

Assuming that the valuation date t is $0 < t < \delta$:

Swap Reversed Cash Flows

$$
\begin{array}{ccccccccc}
 & & & & +A(i_0 - k)\delta & & +A(f_{t,2\delta} - k)\delta & & +A(f_{t,3\delta} - k)\delta \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \text{——} & t & \text{——} & \delta & \text{——} & 2\delta & \text{——} & 3\delta \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\text{contract} & & \text{valuation date} & & \text{reset date} & & \text{reset date} & &
\end{array}
$$

Swap: Value at Time t

$$
\begin{array}{ccccccccc}
 & & swap_{t,0,n,\delta}(k) = & & & & & & \\
 & & A(i_0 - k)\delta B_{t,\delta} & & & & & & \\
 & & +A(f_{t,2\delta} - k)\delta B_{t,2\delta} & & & & & & \\
 & & +A(f_{t,3\delta} - k)\delta B_{t,3\delta} & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \text{——} & t & \text{——} & \delta & \text{——} & 2\delta & \text{——} & 3\delta \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\text{contract} & & \text{valuation date} & & \text{reset date} & & \text{reset date} & &
\end{array}
$$

- $swap_{t,0,n,\delta}(k)$ is the value at t of an n-year δ-reset swap contracted at time $t = 0$ at a strike rate k
- $f_{t,T}$ is the Libor forward rate at time t for delivery at T
- $B_{t,i\delta}$ is the price at t of a bond paying \$1 at $i\delta$

The Black Model for a Caplet

If the BGM process for forward rates holds, the value of a caplet at time t, with maturity T is given by:

$$caplet_{t,T,\delta}(k) = \frac{A}{1+f_{t,T}\delta}\delta[f_{t,T}N(d_1) - kN(d_2)]B_{t,T} \tag{1}$$

where

$$d_1 = \frac{\ln(\frac{f_{t,T}}{k}) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\tau = T - t$$

- $caplet_{t,T,\delta}(k)$ is the value at t of a T-year caplet at a strike rate k
- $f_{t,T}$ is the Libor forward rate at time t for delivery at T
- $B_{t,T}$ is the price at t of a bond paying \$1 at T
- σ is the volatility of the Libor, i_T

The Black Model for Libor Futures Options

Assuming these are European-style, marked-to-market options, then a put on the futures price has a futures value

$$P_{t,T}(k) = [(1 - H_{t,T})N(d_1) - (1 - K)N(d_2)]$$

where

$$d_1 = \frac{\ln\left(\frac{1-H_{t,T}}{1-K}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where $H_{t,T}$ is the futures price and K is the strike price.

The futures price of the option can be established for Libor options by assuming that the futures *rate* follows a lognormal diffusion process (limit of the geometric binomial process as $n \to \infty$).

- $P_{t,T}(k)$ is the value at t of a T-year futures option at a strike rate k
- $H_{t,T}$ is the Libor futures price at time t for delivery at T
- σ is the volatility of Libor

The Black Model for Caplets/Floorlets using Equivalent Bond Options

If the (continuously compounded) interest rate is normally distributed, the price of a floorlet on Libor is given by

$$Floorlet_{t,T,\delta}(k) = \left[B_{t,T+\delta}N(d_1) - KB_{t,T}N(d_2)\right](1 + k\delta)$$

where

$$d_1 = \frac{\ln\left(\frac{F_{t,T,T+\delta}}{K}\right) + \frac{\sigma'^2\tau}{2}}{\sigma'\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma'\sqrt{\tau}$$

$$K = \frac{1}{1 + k\delta}$$

$$\tau = T - t$$

Here, the bond forward price is assumed to follow a lognormal diffusion process.

- $floorlet_{t,T,\delta}(k)$ is the value at t of a T-year floorlet at a strike rate k
- $B_{t,T+i\delta}$ is the price at t of a bond paying \$1 at $T + i\delta$
- σ' is the volatility of the zero-coupon bond price
- K is the strike price of the equivalent bond option

3 Parity Relationships for Interest-Rate Options

Notation

- P_t value of put option at time t
- C_t value of call option at time t
- K strike price
- T option maturity

A long FRA at k is equivalent to $(1 + k\delta)$ short forward contracts on a one-period zero-coupon bond.

A caplet on Libor at k is equivalent to $(1 + k\delta)$ put options on a one-period zero-coupon bond.

An long interest-rate swap (to pay fixed, receive Libor) is equivalent to a short forward contract on an n-year coupon bond, with coupon k.

A European-style payer swaption (to pay fixed, receive Libor) is equivalent to a put option on an n-year coupon bond, with coupon k, and with a strike price of par.