

# The Term Structure of Interest-Rate Futures Prices.<sup>1</sup>

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## Abstract

# The Term Structure of Interest-Rate Futures Prices

We derive general properties of two-factor models of the term structure of interest rates and, in particular, the process for futures prices and rates. Then, as a special case, we derive a no-arbitrage model of the term structure in which any two futures rates act as factors. In this model, the term structure shifts and tilts as the factor rates vary. The cross-sectional properties of the model derive from the solution of a two-dimensional, autoregressive process for the short-term rate, which exhibits both mean-reversion and a lagged persistence parameter. We show that the correlation of the futures rates is restricted by the no-arbitrage conditions of the model. In addition, we investigate the determinants of the volatilities and the correlations of the futures rates of various maturities. These are shown to be related to the volatility of the short rate, the volatility of the second factor, the degree of mean-reversion and the persistence of the second factor shock. We also discuss the extension of our model to three or more factors. We obtain specific results for futures rates in the case where the logarithm of the short-term rate [e.g., the London Inter-Bank Offer Rate (*LIBOR*)] follows a two-dimensional process. We calibrate the model using data from Eurocurrency interest rate futures contracts.

## 1 Introduction

Theoretical models of the term structure of interest rates are of interest to both practitioners and financial academics. The term structure exhibits several patterns of changes over time. In some periods, it shifts up or down, partly in response to higher expectations of future inflation. In other periods, it tilts, with short-term interest rates rising and long-term interest rates falling, perhaps in response to a tightening of monetary policy. Sometimes, its shape changes to an appreciable extent, affecting its curvature. Hence, a desirable feature of a term-structure model is that it should be able to capture shifts, tilts and changes in the curvature of the term structure. In this paper, we present and analyse a model of the term structure of futures rates that has the above properties.

Previous work on the term structure of interest rates has concentrated mainly on bond yields of varying maturities or, more recently, on forward rates. In contrast, this paper concentrates on futures rates, partly motivated by the relative lack of previous theoretical models of interest-rate futures prices. However, the main reason for focussing on futures rates is analytical tractability. Futures prices are simple expectations of spot prices under the equivalent martingale measure (EMM), of the future relevant short-term bond prices, whereas forward prices and spot rates involve more complex relationships. It follows that futures prices and futures rates are fairly simple to derive from the dynamics of the spot rate. In contrast, closed-form solutions for forward rates have been obtained only under rather restrictive (e.g. Gaussian) assumptions. Further, from an empirical perspective, since forward and futures rates differ only by a convexity adjustment, it is likely that most of the time series and cross-sectional properties of futures rates are shared by forward rates, to a close approximation, at least for short maturity contracts. It makes sense, therefore, to analyse these properties, even if the ultimate goal is knowledge of the term-structure behaviour of forward or spot prices. Finally, the analysis of futures rates is attractive because of the availability of data from trading on organized futures exchanges. Hence, the models derived in the paper are directly testable, using data from the liquid market for Eurocurrency interest rate futures contracts.

One contrast between many of the term-structure models presented in the academic literature and those used by practitioners for the pricing of interest rate options is in the distributional assumptions made. Most of the models

which derive the term structure, for example Vasicek (1977), Balduzzi, Das, Foresi and Sundaram (1996) (BDFS), Jegadeesh and Pennacchi (1996) (JP), and Gong and Remolona (1997) assume that the short term interest rate is Gaussian. In contrast, popular models for the pricing of interest-rate options often assume that rates are lognormally distributed (see for example Black and Karasinski (1991), Brace, Gatarek and Musiela (1997), and Miltersen, Sandmann and Sondermann (1997)). In this paper, we present a model where the short-term rate *LIBOR* is assumed to follow a multi-dimensional lognormal process under the EMM. We show that the no-arbitrage futures rates for all maturities are also log-linear in any two (or three) rates. This implies that all futures rates are also lognormal in our model. We then show that the correlation between the long and short maturity futures rates is restricted by the degree of mean-reversion of the short rate and the relative volatilities of the long and short-maturity futures rates. Also, the volatility structure of futures rates of various maturities can be derived explicitly from the assumed process for the spot short rate.

The performance of a set of models where the short rate follows a process with a stochastic central tendency has been analysed in a recent article by Dai and Singleton (2000) (DS). They analyse the set of "affine" term structure models introduced previously by Duffie and Kan (1994). Our process for the short rate is also a stochastic central tendency model. However, there are some differences that should be noted between our model and those analysed by DS. First, in our model the short rate follows a discrete-time, rather than a continuous-time process. Second, as noted above, we assume that the process is log-Gaussian. Third, we assume that the short rate follows the process under the risk-neutral measure; hence, we abstract from considerations of the market price of risk.

We suggest a time series model in which the short rate follows a stochastic central tendency process. This assumption allows us to nest the  $AR(1)$  single-factor model as a special case. It is also general enough to produce stochastic no-arbitrage term structures with shapes that capture most of those observed empirically. A similar model, in which the conditional mean of the short rate is stochastic has been suggested by Hull and White (1994), JP (1996), BDFS (1996) and analysed by Gong and Remolona (1997). The characteristics of these models are reviewed below.

The literature on the pricing of futures contracts was pioneered by Cox, Ingersoll and Ross (1981) [CIR], who characterized the futures price of an

asset as the expectation, under the risk-neutral measure, of the spot price of the asset on the expiration date. In a related paper, Sundaresan (1991) shows that, under the risk-neutral measure, the futures interest rate is the expectation of the spot interest rate in the future. This follows from the fact that the *LIBOR* futures contract is written on the three-month *LIBOR* itself, rather than on the price of a zero-coupon instrument.

Jegadeesh and Pennacchi (1996) [JP] suggest a two-factor equilibrium model of bond prices and *LIBOR* futures based on a two-factor extension of the Vasicek (1977) Gaussian model. They assume that the (continuously compounded) interest rate is normally distributed and generated by a process with a stochastic central tendency. They then estimate the model using futures prices of *LIBOR* contracts, backing out estimates of the coefficients of mean-reversion of the short rate as well as the second stochastic conditional-mean factor. Our approach is somewhat different. We directly assume a process for *LIBOR*, rather than the continuously compounded rate, and then derive the process for the *LIBOR* futures.<sup>1</sup> This allows us to analyse the less tractable log-Gaussian case.

Our general model is closely related to the JP paper, with the important distinction that it is embedded in an arbitrage-free, rather than an equilibrium framework, thus eliminating the need for explicitly incorporating the market price of risk. Although our analysis is based on weaker assumptions, we are able to derive quite general, distribution-free results for futures rates. We then include, as a significant special case, a model in which the interest rate is lognormal.

The work of Gong and Remolona (1997) is similar, in some respects, to that of JP. Like JP, they also employ a two-factor model, in which the second factor is the conditional mean of the short-rate process. However, they focus on the yield of long-dated bonds rather than on the futures rates. In their model, the short-term rate is linear in the two factors. Also, in a manner similar to Vasicek (1977), they assume a market price of risk, solve for bond prices, and back-out the long-term rates and the variances of the two factors.<sup>2</sup>

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<sup>1</sup>This is similar to the approach taken in the *LIBOR* market models of Brace, Gatarek and Musiela (1997), and Miltersen, Sandmann and Sondermann (1997).

<sup>2</sup>Much of the recent literature, dating back to the work of Ho and Lee (1986), has been concerned with the evolution of forward rates. The most widely cited work in this area is by Heath, Jarrow and Morton [HJM] (1990a, 1990b, 1992). HJM provide a continuous-

The two-factor models developed in this paper are related also to the exponential affine-class of term-structure models introduced by Duffie and Kan (1994). This class is defined as the one where the continuously compounded spot rate is a linear function of any  $n$  factors or spot rates. In an interesting special case of our model, where the logarithm of the *LIBOR* evolves as a two-dimensional linear process, it is the logarithm of the futures rate that is linear in the logarithm of any two futures rates.

The remainder of this paper is organised as follows. In section 2, we derive some general properties that characterise two-factor models in general and show that if the function of the price of a zero-coupon bond follows a two-dimensional process, its conditional expectation is generated by a two-factor model. We then derive a two-factor, cross-sectional model for futures prices in the general case where a function of the price of a zero-coupon bond follows a stochastic central tendency process. In section 3, we assume that the logarithm of the London Inter-Bank Offer Rate (*LIBOR*) follows a two-dimensional process and derive our main result for futures contracts on the *LIBOR*: a two-factor cross-sectional relationship between the changes in the prices of interest-rate futures. We estimate the parameters of the two-factor model using estimates of the volatilities and correlations of futures rates derived from the Eurodollar futures contract for the period 1995-99, in section 4. The conclusions and possible applications of our model to the valuation of interest rate options and to risk management are discussed in section 5. Here, we also discuss possible extensions and, in particular, the generalisation of the model to three factors.

## 2 Some general properties of two-factor models

In this section, we establish two statistical results, that hold for any two-factor process of the form that we assume for the short-term rate. These

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time limit to the Ho-Lee model and generalize their results to a forward rate which evolves as a generalized Ito-process with multiple factors. The HJM paper can be distinguished from our paper in terms of the inputs to the two frameworks. The required input to the HJM-type models is the term structure of the volatility of forward rates. In contrast, in our paper, we derive the term structure of volatility of futures rates from a more basic assumption regarding the process for the spot rate. To the extent that the futures and forward volatility structures are related, our analysis in this paper provides a link between the spot-rate models of the Vasicek type and the extended HJM-type forward rate models.

results are used to establish a general proposition, that holds for the conditional expectation of any function of the zero-coupon bond price. The conditional expectation is of key significance, since the futures price (or rate) is closely related to the conditional expectation of the future spot interest rate. Later in the section, these results are directly applied to establish futures prices and rates.

## 2.1 Definitions and notation

We denote  $P_t$  as the time- $t$  price of a zero-coupon bond paying \$1 with certainty at time  $t + m$ , where  $m$  is measured in years. The short-term interest rate is defined in relation to this  $m$ -year bond, where  $m$  is fixed. The continuously compounded short-term interest rate for  $m$ -year money at time  $t$  is denoted as  $i_t$ , where  $i_t = -\ln(P_t)/m$ . The London Interbank Offer Rate *LIBOR*, again for  $m$ -year money is denoted  $r_t$ , where  $P_t = 1/(1+r_tm)$ , where  $m$  is the proportion of a year.<sup>3</sup>

We are concerned with interest rate contracts for delivery at a future date  $T$ . We denote the futures rate as  $F_{t,T}$ , the rate contracted at  $t$  for delivery at  $T$  of an  $m$ -period loan. We denote the logarithm of the *LIBOR* futures rate as  $f_{t,T} = \ln [F_{t,T}]$ . Note that, under this notation, which is broadly consistent with HJM,  $F_{t,t} = r_t$  and  $f_{t,t} = \ln (r_t)$ .

## 2.2 General properties of two-factor models

The two-factor stochastic central tendency models introduced by JP and BDFS have the general form:

$$dx_t = k_1(\theta_t - x_t)dt + \sigma_1 dz_1 \quad (1)$$

$$d\theta_t = k_2(\bar{\theta} - \theta_t)dt + \sigma_2 dz_2 \quad (2)$$

where  $dz_1 dz_2 = \rho_{1,2} dt$ , and where  $\rho_{1,2}$  is the correlation between the Wiener processes  $dz_1$  and  $dz_2$ . The coefficients  $k_1$  and  $k_2$  measure the degree of mean reversion of the variables to their respective means,  $\theta_t$  and  $\bar{\theta}$ .  $\sigma_1$  and

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<sup>3</sup>In the *LIBOR* contract,  $m$  has to be adjusted for the day-count convention. Hence,  $m$  becomes the actual number of days of the loan contract divided by the day-count basis (usually 360 or 365 days).

$\sigma_2$  are the instantaneous standard deviations of the variables  $x$  and  $\theta$ . In JP,  $x_t = i_t$  is the continuously compounded short-term interest rate. The central tendency  $\theta_t$  is interpreted as either a “federal funds target” or as reflecting “investors’ rational expectations of longer-term inflation.” In the paper by BDFS,  $x_t$  is again the short-term interest rate. However, they suggest a slightly more general process for the central tendency

$$d\theta_t = k_2(\bar{\theta} - \theta_t + a)dt + \sigma_2 dz_2 \quad (3)$$

The constant allows for a drift in the central tendency. In both models, the short-term interest rate is normally distributed.

Essentially, these models are two-factor extensions of the one-factor Vasicek (1977) model. A similar extension is made in Hull and White (1994), with a slightly different assumption that allows the drift of the short-term interest rate to be time dependent. Hull and White assume that  $x_t$  is any function of the short rate, for example; if  $x_t = \ln(i_t)$ , where  $i_t$  is the continuously compounded short term rate, then the model is a two-factor extension of the lognormal Black and Karasinski model. Hull and White state the model in a slightly different form from that of JP and BDFS:

$$dx_t = (\phi_t + u_t - k_1 x_t)dt + \sigma_1 dz_1 \quad (4)$$

$$du_t = -k_2 u_t dt + \sigma'_2 dz_2 \quad (5)$$

However, this model can be shown to be a simple transformation of the BDFS model, with the generalisation that the drift term  $a$  is time dependent, i.e. (3) contains a term  $a_t$ .<sup>4</sup>

In this paper, we consider the general Hull-White class of two-factor models. In discrete form, these models have the following general structure:<sup>5</sup>

$$x_{t+1} - x_t = c(\phi_t - x_t) + y_t + \epsilon_{t+1} \quad (8)$$

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<sup>4</sup>Let

$$\phi_t = k_1 \theta_t - u_t, \quad (6)$$

$$\sigma'_2 = k_1 \sigma_2. \quad (7)$$

Then substitution of equations (6) and (7) in equations (4) and (5) yields the general BDFS model (1) and (3).

<sup>5</sup>Dai and Singleton (2000), in the discussion following their equation (20), argue that models of this type, with the additional restriction that the innovations in  $r_t$  and  $\theta_t$  are uncorrelated, fail to capture the empirical term structure of volatility. They suggest that

$$y_{t+1} - y_t = -\alpha y_t + \nu_{t+1}, \quad (9)$$

$$E(\epsilon_{t+1}) = E(\nu_{t+1}) = 0 \quad (10)$$

where

$$c = 1 - e^{-k_1 n}$$

$$\alpha = 1 - e^{-k_2 n},$$

and  $n$  is the length of the time period. In this model,  $y_t$  has an initial value of zero and mean reverts to zero.<sup>6</sup> It follows that  $E(y_t) = 0$ , and hence, taking expectations in equation (8),

$$E(x_{t+1}) - E(x_t) = c(\phi_t - x_t)$$

Then, substituting back in equation (8) we have

$$x_{t+1} - E(x_{t+1}) = (1 - c)[x_t - E(x_t)] + y_t + \epsilon_{t+1} \quad (11)$$

Hence, in this type of model, the value of  $x_t$  reverts at the rate  $c$  to its initial expected value.

We now establish that, if a variable follows the above process, the conditional expectation of the variable is necessarily governed by a two-factor cross-sectional model. We begin by proving this result quite generally, and then apply it to the special case where  $x_t$  is an interest rate or its logarithm.

**Lemma 1** *The variable  $x_t$  follows the process*

$$x_{t+1} - E(x_{t+1}) = (1 - c)[x_t - E(x_t)] + y_t + \epsilon_{t+1}$$

where

$$y_{t+1} - y_t = -\alpha y_t + \nu_{t+1}$$

if and only if the conditional expectation of  $x_{t+k}$  is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

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humped shaped volatility term structures are not possible in the above model and the volatility structure is monotonic. However, as shown, for example, by Hull and White (1994), Figure 3, this is not true. The simulations in Section 4 below also illustrate this point in some detail for the case of futures rate volatilities.

<sup>6</sup>See Hull and White (1994), footnote 4.

where

$$b_k = \sum_{\tau=1}^k (1-c)^{k-\tau} (1-\alpha)^{\tau-1} \quad (12)$$

$$a_k = (1-c)^k - (1-c)b_k. \quad (13)$$

*Proof.* See appendix 1.

First, let us take the simplest case where  $x$  is the short-term rate of interest. The lemma then implies that if the short rate follows the Hull-White process, the expectation of the short rate  $k$  periods hence, is linear in the short rate and the expectation of the short rate at  $t = 1$ . The implication of this result can be illustrated as follows. Under the expectations hypothesis, where the forward rate is the expectation of the future spot rate, it follows that the  $k$ -th period forward rate is a linear function of the spot rate and the  $t = 1$  forward rate.

One important implication of the lemma is that in this type of two-factor model, the cross-sectional linear coefficients  $a_k$  and  $b_k$  are invariant to the interchange of the mean reversion coefficient,  $c$  and the persistence parameter,  $\alpha$ . That is

$$a_k, b_k | c, \alpha = a_k, b_k | c', \alpha',$$

if  $c' = \alpha$ , and  $\alpha' = c$ . This identification problem means that it may be difficult to distinguish models where the interest rate reverts very rapidly to a slowly decaying central tendency, from those where the interest rate reverts slowly to a rapidly decaying central tendency.<sup>7</sup>

Lemma 1 states the implications of a two-dimensional, stochastic conditional mean, process for an arbitrary function of the zero-coupon bond price. The function could be a rate of interest, such as the continuously compounded rate (as in HJM) or the *LIBOR* (as in BGM), or it could be any other function of the price of the zero-coupon bond itself. The lemma restricts the cross-sectional properties of the conditional expectation. As we will

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<sup>7</sup>This partially explains why, in the calibration of the model in section 4, there are two sets of parameters that yield exactly the same volatilities and correlations of the futures rates. It may also explain discrepancies in empirical results presented in JP, DS and others.

see in the following section, these properties are directly relevant for the investigation of futures prices and rates. The intuition behind Lemma 1 is that the two influences on the function of the zero-bond prices, one of which is lagged, yield a cross-sectional structure with two factors.

### 2.3 Interest-Rate Futures Prices in a No-Arbitrage Economy

In this sub-section, we apply the results in the previous sub-section to derive futures prices and futures interest rates in a no-arbitrage setting. We assume here that the two-dimensional process, specified in equation (11) for  $x_t = f(P_t)$ , holds under the Equivalent Martingale Measure (EMM). The EMM is the measure under which all zero-coupon bond prices, normalised by the money market account, follow martingales. Assuming that the process followed by  $x_t$  is of the Hull-White type under the risk-neutral measure abstracts from considerations of the market price of risk. It allows us to directly derive prices for futures contracts on  $x_t$ .

Cox, Ingersoll and Ross (1981) and Jarrow and Oldfield (1981) established the proposition that the futures price, of any asset, is the expected value of the future spot price of the asset, where the expected value is taken with respect to the equivalent martingale measure. We can now apply this result to determine futures prices, assuming that a function of the bond price is generated by the two-dimensional process analysed above. Since there is a one-to-one relationship between zero-coupon bond prices and short-term interest rates, defined in a particular way, we can then proceed to derive a model for futures interest rates.

Initially, we make no specific distributional assumptions. We assume only a) a no-arbitrage economy in which the EMM exists, b) that  $x_t$ , which is some function of the time  $t$  price of an  $m$ -year zero-coupon bond, follows a stochastic central tendency process of the general form assumed in Lemma 1, and c) that a market exists for trading futures contracts on  $x_t$ , where the contracts are marked-to-market at the same frequency as the definition of the discrete time-period from  $t$  to  $t + 1$ . We first establish a general result, and then illustrate it with some familiar examples. We denote the futures price, at  $t$ , for delivery of  $x_{t+k}$  at  $t + k$ , as  $x_{t,k}$ . We now have:

**Proposition 1** (*General Cross-Sectional Relationship for Futures Prices*)

Assume that equation (11) holds for  $x_t$  under the EMM, where  $x_t = f(P_t)$ , then the  $k$ -period ahead futures price of  $x_t$  is given by the two-factor linear relationship:

$$x_{t,k} - x_{0,k} = a_k [x_t - x_{0,t}] + b_k [x_{t,t+1} - x_{0,t+1}]$$

where

$$b_k = \sum_{\tau=1}^k (1-c)^{k-\tau} (1-\alpha)^{\tau-1} \quad (14)$$

$$a_k = (1-c)^k - (1-c)b_k. \quad (15)$$

*Proof.* From CIR (1981), proposition 2 and Pliska (1997), the futures price of any payoff is its expected value, under the EMM. Applying this result to  $x_{t+k}$ , yields

$$x_{t,k} = E_t(x_{t+k}) \quad (16)$$

Substituting equation (16) in Lemma 1 yields the statement in the Proposition.  $\square$

The rather general result in Proposition 1 is of principal interest because of two special cases. The first is the case where the futures contract is on the zero-coupon bond itself. The second is the case of a futures contract on an interest rate, such as *LIBOR*, which is a function of the zero-coupon bond price. The first case is typified by the T. Bill futures contract traded on the Sydney Futures Exchange. The second case is the most common, typified by the Eurocurrency futures contract. We consider these cases in the corollaries below.

### Remark

An interesting implication of Proposition 1 is that if the bond price itself follows a linear process, under the EMM, then the futures prices of the bond will be given by a two-factor cross-sectional relationship. This follows by assuming that  $x_t = f(P_t) = P_t$  in the Proposition. This shows that futures prices at time  $t$  are generated by a linear two-factor model, if and only if the zero-bond price follows a process of the Hull-White type. Note that the two factors generating the  $k$ th futures price are the spot price of the bond and the first futures price, i.e., the futures with maturity equal to  $t + 1$ .

Similarly, the variance of the  $k$ th futures price is determined by the variance of the spot bond price, the variance of the conditional mean and the mean-reversion coefficients. This is helpful in understanding the conditions under which the term structure follows a two-factor process. Essentially, if futures prices of long-dated futures contracts are given by the cross-sectional model in Proposition 1, then forward prices, and also futures and forward rates will follow two-factor models. The relationship for interest rates, however, is, in general, complex, since the functions  $i_t(P_t)$  and  $r_t(P_t)$  are non-linear.

We now illustrate the use of Proposition 1 in the case of interest rate (as opposed to bond-price) futures. Instead of assuming that the price of a zero-coupon bond follows a two-dimensional, linear process, we now assume that the interest rate, *LIBOR*, follows a stochastic central tendency process. We have the following corollary of Proposition 1:

**Corollary 1** (*Cross-Sectional Relationship for LIBOR Futures Prices for the Case of a Linear Process for the LIBOR*)

*In a no-arbitrage economy, the short-term rate of interest follows a process of the form*

$$r_{t+1} = E_0[r_{t+1}] + (1 - c)[r_t - E_0(r_t)] + y_t + \epsilon_{t+1} \quad (17)$$

where

$$y_{t+1} - y_t = -\alpha y_t + \nu_{t+1}$$

and

$$E_t(\epsilon_{t+1}) = 0, \quad E_t(\nu_{t+1}) = 0,$$

*if and only if the term structure of futures rates at time  $t$  is generated by a two-factor model, where the  $k$ th futures rate is given by*

$$\begin{aligned} r_{t,k} - r_{0,t+k} &= a_k[r_t - r_{0,t}] \\ &+ b_k[r_{t,t+1} - r_{0,t+1}] \end{aligned} \quad (18)$$

where  $a_k$  and  $b_k$  are given by (15) and (14) respectively.

*Proof.* The proof of the corollary again follows as a special case of Proposition 1, where  $x_t = r_t$ .  $\square$

The corollary illustrates the simple two-factor structure of futures rates that is implied by the two-dimensional process for the spot rate. Note that the mean-reversion coefficients are embedded in the cross-sectional coefficients,  $a_k$  and  $b_k$ . Also, given the linear structure, it follows from (18), that the futures rates will be normally distributed, if the spot rate and the first futures rate are normally distributed. Hence, the corollary could be helpful in building a Gaussian model of the term structure of futures rates.<sup>8</sup>

### 3 *LIBOR* futures prices in a lognormal short-rate model

In the previous section, we showed that if the short-term *LIBOR* interest rate, evolves as a two-dimensional mean-reverting process under the risk-neutral measure, then a simple cross-sectional relationship exists between futures rates of various maturities. In principle, this type of model could be applied to predict relationships between the prices of Eurocurrency futures contracts, based on *LIBOR* or some other similar reference rate, which are the most important short-term interest rate futures contracts traded on the markets. However, in the case of *LIBOR*, the consensus in the academic literature and in market practice is that changes in interest rates are dependent on the level of interest rates. In particular, a lognormal distribution for short-term interest rates is commonly assumed.<sup>9</sup>

When the logarithm of the short-term interest rate follows a linear process, the results of the analysis of futures prices in section 2 cannot be used directly, since the market does not trade futures on the logarithm of the *LI-*

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<sup>8</sup>In a two-factor extension of the Vasicek (1977) framework, JP (1996) estimate a two-factor term structure model that is similar to that in equation (18) under the assumption of normally distributed interest rates. They show that their model fits the level of Eurodollar short-term interest rates contracts rather well for maturities of up to two years, and *changes* in the rates for longer-dated contracts. It is possible that this is because of ignoring the skewness effect (due to the normality assumption), which becomes significant for longer-dated contracts.

<sup>9</sup>This is borne out by the empirical research of Chan et.al.,(1992) and more recently of Eom (1994) and Bliss and Smith (1998). There continues to be debate over the elasticity parameter of the changes in interest rates with respect to the level.

*BOR*. However, if it is assumed that the *LIBOR* follows a lognormal process, standard results relating the mean of the lognormal variable to its logarithmic mean can be used to derive results for futures prices in this case.

We assume now that the logarithm of the *LIBOR* follows a stochastic central tendency process, under the equivalent martingale measure. First, we introduce the following notation. The mean and annualised standard deviation at time  $t$  (of the logarithm) of the spot rate at time  $T$ , under the EMM, are denoted

$$\begin{aligned}\mu(t, T, T) &= E_t[f_{T,T}] \\ \sigma(t, T, T) &= [\text{var}_t[f_{T,T}]/(T-t)]^{\frac{1}{2}}\end{aligned}$$

respectively. In the case of futures rates, we define the mean and standard deviation at time-0 of the log-futures at time- $t$  for delivery at time- $T$  as

$$\begin{aligned}\mu(0, t, T) &= E_0[f_{t,T}] \\ \sigma(0, t, T) &= [\text{var}_0[f_{t,T}]/t]^{\frac{1}{2}}\end{aligned}$$

Table 1 summarizes the notation used in the paper.

Using Lemma 1, we have

**Corollary 2** (*Cross-Sectional Relationship for the Change in Expected Values for the Case of the Logarithmic Spot Rates*)

*The logarithm of the spot rate follows the process*

$$f_{t+1,t+1} - \mu(0, t+1, t+1) = [f_{t,t} - \mu(0, t, t)](1-c) + y_t + \epsilon_{t+1}, \quad \forall t$$

where

$$y_{t+1} = y_t(1-\alpha) + \nu_{t+1}$$

if and only if the expectation of the logarithm of the interest rate  $r_{t+k}$  at time  $t$  is

$$\begin{aligned}\mu(t, t+k, t+k) - \mu(0, t+k, t+k) &= a_k[f_{t,t} - \mu(0, t, t)] \\ &+ b_k[\mu(t, t+1, t+1) - \mu(0, t+1, t+1)]\end{aligned}$$

*Proof.* The corollary follows directly from Lemma 1, where  $x_t$  is any function of  $P_t$ .  $\square$

In this case, the spot rate follows a logarithmic, mean-reverting process with a stochastic conditional mean. The implication is that the conditional expectation of the logarithmic rate for maturity  $t + k$  is generated by a two-factor cross-sectional model. The corollary has direct implications for the behaviour of futures rates in a logarithmic short-rate model.

We now make use of the following lemma:

**Lemma 2** (*Lognormal Futures Rates*)

*In a no-arbitrage economy, if the LIBOR rate follows a lognormal process under the equivalent martingale measure, then*

a) *the  $k$ -period LIBOR futures rate at time  $t$  is*

$$F_{t,t+k} = \exp\left[\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)\right]$$

*where  $n$  is the length, in years, of the period  $t$  to  $t + 1$ .*

b) *Also, the  $k$ -period LIBOR futures rate at time  $t$ ,  $F_{t,t+k}$  is lognormal, with logarithmic mean*

$$\mu(0, t, t+k) = \mu(0, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)$$

*We also have:*

$$\mu(t, t+k, t+k) = f_{t,t+k} - \frac{kn}{2}\sigma^2(t, t+k, t+k),$$

$$\mu(0, t+k, t+k) = f_{0,t+k} - \frac{kn}{2}\sigma^2(0, t+k, t+k),$$

*Proof.* See Appendix 2.

Lemma 2 establishes first that the lognormality of the spot *LIBOR* implies lognormality of the *LIBOR* futures. This is important for our analysis of the

behaviour of the futures rate, in this section. This property follows from the CIR (1981) result that the futures price is the expectation, under the EMM, of the spot price. Second, the lemma establishes a useful relationship between the logarithmic mean of the futures rate and that of the corresponding spot rate.

We use this relationship, which itself follows from the lognormality of the futures and spot rates, in the proof of the following proposition. We then have:

**Proposition 2 (Cross-Sectional Relationship between Futures Rates)**

*Consider a no-arbitrage economy, in which the LIBOR rate follows a two-dimensional lognormal process, under the equivalent martingale measure, of the form*

$$f_{t+1,t+1} = \mu(0, t+1, t+1) + [f_{t,t} - \mu(0, t, t)](1 - c) + y_t + \epsilon_{t+1}$$

where

$$y_{t+1} = (1 - \alpha)y_t + \nu_{t+1},$$

and where  $\epsilon_t$  and  $\nu_t$  are normally distributed variables, i.e., the term structure of futures rates at time  $t$  is generated by a two-factor model.

Then, the  $k$ th futures rate is given by

$$\begin{aligned} f_{t,t+k} - f_{0,t+k} &= M_k + a_k[f_{t,t} - f_{0,t}] \\ &\quad + b_k[f_{t,t+1} - f_{0,t+1}] \end{aligned}$$

where  $a_k$  and  $b_k$  are given by (15) and (14) respectively, and  $M_k$  is defined as follows:

Also, using Lemma 2 b), we can write the drift term in terms of the futures volatilities:

$$\begin{aligned} M_k &= -\frac{nt}{2}\sigma^2(0, t, t+k) \\ &\quad + a_k\left[\frac{tn}{2}\sigma^2(0, t, t)\right] \\ &\quad + b_k\left[\frac{nt}{2}\sigma^2(0, t, t+1)\right] \end{aligned}$$

*Proof.* From Corollary 2,

$$\begin{aligned} \mu(t, t+k, t+k) - \mu(0, t+k, t+k) &= a_k[f_{t,t} - \mu(0, t, t)] \\ &+ b_k[\mu(t, t+1, t+1) - \mu(0, t+1, t+1)] \end{aligned}$$

is a necessary and sufficient condition. Substituting the results of Lemma 2 then yields the statement in the proposition.  $\square$

Proposition 2 is the main result of this paper. The Proposition shows the conditions under which a simple log-linear relationship exists for futures rates of various maturities. In this cross-sectional model, futures rates are related to the spot *LIBOR* and the first *LIBOR* futures. The result extends to the lognormal *LIBOR* case the prior results on the term structure of Duffie and Kan (1993). Proposition 2 relates the  $k$ th futures rate, (i.e., the one expiring in  $k$  periods) to the spot rate  $f_{t,t}$  and the first futures rate,  $f_{t+1,t+1}$ . For example, this means that the  $k$ th three-month futures rate is related to the spot three-month rate and the one-period, three-month futures rate. However, following Duffie and Kan (1993), if the model is linear in two such rates, it can always be expressed in terms of *any* two futures rates. In the present context, therefore, the  $k$ th futures rate can be expressed as a function of the spot rate and the  $N$ th futures rate, for example. We have the following implication of Proposition 2:

**Corollary 3** *Suppose any two futures rates are chosen as factors, where  $N_1$  and  $N_2$  are the maturities of the factors, then the following linear model holds for the  $k$ th futures rate:*

$$\begin{aligned} f_{t,t+k} &= f_{0,t+k} + A_k(N_1, N_2)[f_{t,t+N_1} - f_{0,t+N_1}] \\ &+ B_k(N_1, N_2)[f_{t,t+N_2} - f_{0,t+N_2}] \end{aligned} \quad (19)$$

where

$$B_k(N_1, N_2) = (a_k b_{N_1} - b_k a_{N_1}) / (a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1, N_2) = (-a_k b_{N_1} + b_k a_{N_1}) / (a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

and where  $a_k$  and  $b_k$  are defined as before.

*Proof.* Corollary 3 follows by solving equation (18) for  $k = N_1$ , and  $k = N_2$  and then substituting back into equation (18).  $\square$

The  $k$ th futures rate is log-linear in any two futures rates. The meaning of the result is illustrated by the following special case, where there is no mean-reversion in the short rate, i.e., the logarithm of the *LIBOR* follows a random walk.

#### Corollary 4 The Random Walk Case

*Suppose that  $c = 0$ , i.e., the logarithm of the LIBOR follows a random walk. In this case, the  $k$ th futures LIBOR is*

$$f_{t,t+k} - f_{0,t+k} + \left(\frac{N-k}{N}\right) [f_{t,t} - f_{0,t}] + \left(\frac{k}{N}\right) [f_{t,t+N} - f_{0,t+N}]. \quad (20)$$

*Proof.* Corollary 4 follows directly from Corollary 3 with

$$b_{k,N} = \frac{k}{N},$$

and hence,

$$a_{k,n} = \frac{N-k}{N}.$$

$\square$

Here, the  $k$ th futures is affected by changes in the  $N$ th futures according to how close  $k$  is to  $N$ . Equation (20) is a simple two-factor “duration-type” model, in which the term structure of futures rates shifts and tilts.

## 4 LIBOR Futures, Volatilities and Correlation

The model developed in the previous section relates the futures rate, with maturity  $k$ , to the spot *LIBOR*. The implication of Proposition 2 is that the variance of the  $k$ -th futures rate and the correlation of the  $k$ -th futures rate with the spot rate are also determined by the model. In this section, we derive the explicit implications of the stochastic central tendency lognormal spot-rate process for the volatility and correlation structure of the futures rates. We first derive expressions for the volatility and correlation of the futures rates, given the correlation of the two stochastic factors in the model. We then calibrate the model using empirical estimates of the volatilities and the correlations estimated from *LIBOR* futures data for the period 1995-9. Finally, we simulate the calibrated model and illustrate the output of the model in relation to the data.

It follows directly from the result in Proposition 2 that the conditional (log) variance of the  $k$ -th futures is

$$\begin{aligned} \text{var}_{t-1}(f_{t,t+k}) &= a_k^2 \text{var}_{t-1}(f_{t,t}) + b_k^2 \text{var}_{t-1}(f_{t,t+1}) \\ &+ 2a_k b_k \text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) \end{aligned} \quad (21)$$

where

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) = (1-c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(\epsilon_t, \nu_t). \quad (22)$$

It now follows that:

**Proposition 3** (*The Volatilities and Correlations of Futures Rates*)

*Suppose the logarithm of the spot rate follows the process in Corollary 2. The variance of the  $k$ -th futures rate is given by*

$$\begin{aligned} \sigma^2(t-1, t, t+k) &= a_k^2 \sigma^2(t-1, t, t) + b_k^2 \sigma^2(t-1, t, t+1) \\ &+ 2a_k b_k [(1-c) \sigma^2(t-1, t, t) \\ &+ 2\rho \sigma(t-1, t, t) \sigma_\pi] \end{aligned} \quad (23)$$

*and the correlation of the spot and futures rates is therefore given by:*

$$\rho(t-1, t, t+k) = \frac{(1-c)^k \sigma^2(t-1, t, t) + b_k \rho \sigma(t-1, t, t) \sigma_\pi}{\sigma(t-1, t, t) \sigma(t-1, t, k)} \quad (24)$$

where

$$\sigma^2(t-1, t, t+1) = (1-c)^2\sigma^2(t-1, t, t) + \sigma_\pi^2 + 2(1-c)\rho\sigma_\pi\sigma(t-1, t, t)$$

and where  $\rho$  is the correlation between the errors  $\epsilon_t$  and  $\nu_t$ .  $\sigma_\pi$  is the volatility of  $\pi$ .

*Proof.* See Appendix 3.

In Proposition 3, the volatility of the  $k$ -th futures rate is determined by the mean-reversion of the short rate, the variance of the short rate, the mean-reversion and variance of the stochastic mean factor, and the correlation  $\rho$  of the innovations in the two factors.

The coefficient of correlation between the futures rates and the spot rate is important for two reasons. First, the correlation between any two futures rates, which may be taken as factors in the above model, cannot be determined independently of the mean-reversion of the short rate,  $c$ , and the persistence of the conditional mean shock factor,  $\alpha$ . Second, the correlation is an important determinant of the value of certain derivatives, whose payoff depends on the difference between various rates of interest in the term structure.

This expression for the correlation of the short rate and the  $k$ th futures rate illustrates an important implication of the no-arbitrage model. Given the volatilities of the spot and futures rates and the correlation of the errors, we cannot independently choose *both* the correlation and the degree of mean-reversion. The no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean-reversion of the short rate.<sup>10</sup> Further, because the futures volatility depends, in addition, on the degree of persistence of the premium factor shock, all the parameters affect the correlation.

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<sup>10</sup>This would imply that one cannot arbitrarily specify a two-factor model such as Brennan and Schwartz (1979) or Bühler et.al., (1999) without restricting the correlation between the short and long rates.

#### 4.1 Calibration of the Model and the Term Structure of Volatilities and Correlations of Futures Interest Rates

In order to calibrate our model, we construct a time series of futures rates from historical data on spot *LIBOR* and Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, for all available contracts. (The futures rate is defined as 100 minus the futures price.) During the sample period, the first 20 contracts, going out to 5 years at quarterly intervals, were reasonably liquid. The choice of the sample period was dictated by the fact that the Eurodollar futures market became liquid for longer-dated contracts only in the mid-nineties. Furthermore, several studies have documented poor liquidity and inefficiencies, in the Eurodollar futures market, in earlier periods.<sup>11</sup> This was also confirmed by our own analysis using data from the previous period, January 1, 1990 to December 31, 1994.<sup>12</sup>

Since the futures maturity dates are fixed, we interpolate between two adjacent contracts to construct a time-series of futures prices and rates for fixed maturity periods. For instance, if, on a particular trading day, the first contract matures in 63 days and the next one in 154 days, we obtain the futures rate for a maturity of 3 months (91 days), by linearly interpolating between the two futures rates for maturities on either side of 91 days. In this manner, we obtained the futures rates for maturities ranging from 3 months to 60 months.

We first construct the correlogram between the logarithm of the  $t$ th futures rate (maturity of 0, 3, 6, ..., 60 months) and the logarithms of the  $\tau$ th futures rates (maturities of 0, 3, 6, ..., 60 months), as well as the vector of annualized volatilities during the sample period. Table 2 presents annualised volatilities and the correlation matrix, based on quarterly observations of the logarithm of futures rates.<sup>13</sup>

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<sup>11</sup>See, for example, Jegadeesh and Pennacchi (1996) and Gupta and Subrahmanyam (2000).

<sup>12</sup>During this earlier period, only 14 contracts going out to 3.5 years, could be considered reasonably liquid, which would be too small a sample for our calibration exercise.

<sup>13</sup>More frequent samples of the data, such as monthly, weekly and daily, proved to be too noisy, especially for the near-term futures contracts, and resulted in lower correlations. The noise is probably due to trading frictions, including discreteness, non-synchronicity of observations, as well as errors introduced by the interpolation procedure to obtain constant maturity period futures prices and rates.

We then used the estimates of the correlation coefficients as well as their annualized volatilities to calibrate the model. This calibration is implemented by minimising the root mean squared error (RMSE) between the empirical estimates and the values of the volatilities and correlations produced by the two-factor model, where the model values are computed using equations (23) and (24) in Proposition 4.<sup>14</sup>

Since there are two equations to be fitted, the optimization could be achieved by placing all the weight on the first equation (the one for the annualized volatilities of the futures rates) or the second (the equation for the correlation coefficients between the futures rates) or on some weighted combination of the two. In Table 3 we present the resultant parameter estimates and the RMSE for two sets of weights: first, where all the weight is placed on the volatility errors, and second, where equal weight is placed on the volatility errors and the correlation errors. In Panel A of the table we show parameter estimates for the case where the correlation between the factors in the model,  $\rho$ , is constrained to be zero. In panel B we allow the optimisation procedure to choose the optimal  $\rho$  as well as the other parameters.

The results in panel A show that the volatility of the second factor is reduced from 11.20% to 8.66% when the correlations as well as the volatilities are fitted. Also the persistence of the second factor is reduced. The results in panel B reveal only a slight improvement in fit when the correlation of the factors is not constrained to be zero. (The overall RMSE falls marginally from 0.110 to 0.108, in the case where the model is fitted to both volatilities and correlations.) Note also that there are two values of the relevant parameters that yield identical RMSE estimates, shown in the table as solution 1 and solution 2. This is consistent with the previous noted fact that in the model the coefficients  $a_k$  and  $b_k$  are not uniquely determined. The mean reversion  $c$  and the persistence parameter  $\alpha$  can be interchanged without affecting  $a_k$  and  $b_k$ . For example, in the case where the fit is to volatilities and correlations we have: in solution 1  $c = 0.040$ ,  $\alpha = 0.370$ , and in solution 2  $c = 0.370$ ,  $\alpha = 0.040$ . Both solutions yield an identical RMSE of 0.108.

The results of the parameter estimation in Table 3 suggest that correlation of the factors,  $\rho$ , has relatively little effect on the goodness of fit of the model. In order to confirm this, we performed a series of constrained RMSE

<sup>14</sup>The non-linear minimisation of the RMSE was computed using the algorithm of Lason et.al. (1978)

minimisations, using a range of fixed values of  $\rho$ . The results are shown in Table 4, for the case where equal weight is given to the volatilities and the correlations. The RMSE reported in the table shows that the goodness of fit is affected only marginally in the range of  $\rho = -0.3$  to  $\rho = +0.3$ .

We use the estimated values from the calibration exercise to fit the model and derive the term structure of volatilities and correlations. These, along with the actual historical estimates are plotted in Figures 1 and 2. Figure 1 shows the term structure of annualized volatilities of the logarithms of the futures rates of different maturities, computed from the model using the parameter values from the calibration, using only the volatilities (the solid line) and using both the volatilities and correlation coefficients (the line of crosses). The third line shows the graph of the actual volatilities, estimated directly from the data. The figure shows that the model values track the actual values fairly closely. In both cases, the figures show a hump-shaped volatility term structure that has been documented in previous studies.<sup>15</sup>

Figure 2 presents the term structure of correlation coefficients between the logarithms of the futures rates of different maturities. Again, the figure shows that the model values (the line of crosses) track the actual values (the line of circles) fairly closely, when the model is calibrated jointly to volatilities and correlations. In both cases, the correlations are higher for contracts that are closer to each other in maturity. However, as the solid line in the figure indicates, the calibration of the model to volatilities alone produces correlations which are much lower than those estimated from the data.

The humped shape of the term structure of volatility, illustrated in Figure 1, is the result of the volatility of the second factor,  $\sigma_\pi$ , and the mean reversion of the *LIBOR*,  $c$ . The magnitude of the hump depends on the relative size of these parameters. The position of the hump, i.e., the maturity at which the volatility peaks also depends on the persistence parameter,  $\alpha$ . The higher is alpha, the less is the persistence of the second factor and the quicker is the peak reached, given the other parameters. Note that a one-factor model *cannot* produce a humped volatility structure (unless the conditional volatility,  $\sigma_r$  is non-constant). However, a hump can be produced even if the two factors in the model (*LIBOR* and its central tendency) are uncorrelated. This contradicts one of the conclusions of DS. The correlation between the

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<sup>15</sup>See, for example DS and in the Gaussian case Hull and White (1994)

various maturity futures is also determined by the relative size of the parameters. If  $\sigma_\pi$  is small, the spot and futures rates must be highly correlated. However, the mean reversion and persistence parameters are determinants of the correlation of the longer maturity rates. Also, the correlation of the factors themselves,  $\rho$ , is also an important determinant of the correlation structure.

## 4.2 Term Structures of *LIBOR* Futures Produced by the Model

What patterns of term structures of futures rates could be produced by the two-factor model derived in Proposition 2? According to equation (2), the mean-reversion coefficient ( $c$ ), the persistence parameter ( $\alpha$ ), and the correlation coefficient ( $\rho$ ), together with the changes in the short rate and the conditional mean factor should determine the cross-sectional shape of the term structure. In Figure 3, we show an example of the type of term structures which could result, given the two-factor model calibrated to the data as above. We choose the parameter values from solution 1 of Table 3, in the case where the model was fitted to volatilities and correlations. In this case,  $c = 0.04$  and  $\alpha = 0.37$ , and the *LIBOR* mean reverts relatively slowly to a premium factor which itself mean reverts rather slowly. In Figure 3, we assume the initial term structure is flat (futures rates for all maturities are 5%). We then look at the effect of a 1% change, up or down, in the spot rate. In the one-factor model, mean reversion causes the term structure of futures rates to be continuously downward-sloping or continuously upward-sloping, as illustrated by the lines of crosses and circles in the Figure. However in the two-factor model the term structures produced by the model are more complex. In Figure 3, we assume that the stochastic mean also moves up or down in the same direction as the spot *LIBOR* change, but by 0.3%. With the estimated mean reversion parameters the term structure produced has a U-shape or an inverted U-shape, as illustrated by the dashed and solid lines in the figure.

## 5 Conclusions and Extensions

There is a close relationship between the time-series process followed by the short-term interest rate and the cross-sectional characteristics of term

structure models. This paper has explored this relationship in the context of futures prices and futures interest rates. If we assume that the price of a zero-coupon bond (or, indeed, any function of the price) follows a stochastic central tendency process, then no-arbitrage restrictions imply that the term structure of futures prices or rates can be represented by a two-factor cross-sectional model. In an important special case, if we assume that the logarithm of the *LIBOR* interest rate follows a stochastic central tendency, mean-reverting process under the equivalent martingale measure, the term structure of futures rates can be written as a log-linear function of any two rates. The coefficients of this two-factor model are determined by the rates of mean-reversion of the two factors generating the time-series process of the *LIBOR*.

Perhaps the most important theoretical implications of the paper concern the relationship between HJM-type forward rate models and Vasicek-Hull and White-type models of the spot rate process. We have shown in particular that the degree of persistence of the second, conditional mean, factor shock is a critical determinant of the futures-volatility structure. Given the close relationship of futures and forward rates, it must also be an important determinant of the forward-volatility structure, which is an input to the HJM-type models. The well-documented humped volatility structure has been reproduced in our two-factor model with mean-reversion of the short rate and persistence of the conditional-mean factor shock, even in the case where correlations between the innovations in the two factors are zero.

The results in the paper also have some interesting empirical implications. Mean-reversion of short term interest rates is a crucial determinant of the pricing of interest rate contingent claims, in general, and interest rate caps, floors and swaptions, in particular. It is well-known that it is extremely difficult to estimate the coefficient of mean-reversion of short term interest rates from historical data, due to low power. Our model provides an alternative method of estimating the mean-reversion and persistence factors using futures rather than spot data, and using both cross-sectional and time-series data rather than time-series data alone. This derives from the fact that the mean-reversion coefficient, together with the volatility and persistence of the second factor, determines the shape of the futures volatility curve. Hence, observation of the futures volatility curve could lead to improved estimation of mean-reversion. In addition, the model provides the inputs required to judge when a two-factor model may substantially change the pricing and

hedging of interest rate contingent claims, and when a one-factor model is sufficient. Empirical analysis of futures rate volatilities and correlations, using data for the period 1995-99, provides some initial support for the two-factor model.

The two-factor model has the characteristic that any futures rate can be written as a log-linear function of any other two futures rates. The restriction to two stochastic variables and the assumption of lognormal *LIBORs* are important. The restriction to two factors can be relaxed, however, with some cost due to the increased complexity of the model. In Appendix 4, we show that if the second factor  $y_t$  is itself generated by a two-factor model, then a three-factor cross-sectional relationship results. Such a three-factor model nests the two-factor model analysed here and provides a natural generalisation. Another possible generalisation would consider non-lognormal processes with, for example, stochastic volatility, generated perhaps with a GARCH process. Recent empirical findings in Brenner, et al. (1996) suggest that the short-term interest rate follows a stochastic volatility process. We leave such extensions for subsequent research.

## Appendix 1: Properties of the conditional mean for two-dimensional time-series processes (Proof of Lemma 1)

### Lemma 1

The variable  $x_t$  follows the process

$$x_{t+1} - E(x_{t+1}) = (1 - c)[x_t - E(x_t)] + y_t + \epsilon_{t+1}$$

where

$$y_{t+1} - y_t = -\alpha y_t + \nu_{t+1}$$

if and only if the conditional expectation of  $x_{t+k}$  is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

where

$$b_k = \sum_{\tau=1}^k (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} \quad (25)$$

$$a_k = (1 - c)^k - (1 - c)b_k. \quad (26)$$

*Proof.*

### Sufficiency

Successive substitution  $x_1, x_2, \dots, x_{t+k}$  and taking the conditional expectation yields

$$E_t(x_{t+k}) = x_t(1 - c)^k + \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1 - \alpha)^\tau \cdot \sum_{\tau=1}^k (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} \quad (27)$$

Substituting the corresponding expression for  $E_t(x_{t+1})$  :

$$E_t(x_{t+1}) = x_t(1 - c) + \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1 - \alpha)^\tau$$

yields

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}). \quad (28)$$

where

$$b_k = \sum_{\tau=1}^k (1-c)^{k-\tau} (1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

### Necessity

Assume that

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}) \quad (29)$$

where  $a_k$  and  $b_k$  are defined by (28) above, and  $x_t$  and  $E_t(x_{t+1})$  are not perfectly correlated. Consider the orthogonal component  $z_t$  from

$$E_t(x_{t+1}) = \gamma x_t + z_t \quad (30)$$

Then

$$E_t(x_{t+1}) = (a_1 + b_1 \gamma) x_t + b_1 z_t$$

and hence, since  $a_1 = 0$  and  $b_1 = 1$

$$x_{t+1} = \gamma x_t + z_t + \epsilon_{t+1}. \quad (31)$$

where  $E_t(\epsilon_{t+1}) = 0$ .

Hence  $x_t$  follows a two-dimensional process with innovations  $z_t, \epsilon_{t+1}$ .

We now show that  $\gamma = (1-c)$  and also that  $z_t$  follows a mean-reverting process with mean-reversion  $\alpha$ . Suppose by way of contradiction, that  $\gamma = (1-c')$ . Also, suppose there is a shock such that  $x_t$  changes while the difference,  $E_t(x_{t+1}) - x_t$ , is constant; then,  $E_t(x_{t+k})$  will not be given by equation (28), since  $c \neq c'$ . It follows that we must have  $\gamma = (1-c)$ . Second, suppose that  $\gamma = (1-c)$ , but  $z_t$  mean-reverts at a rate different from  $\alpha$ . Then, if the difference,  $E_t(x_{t+1}) - x_t$ , changes, while  $x_t$  is constant, then again  $E_t(x_{t+k})$  will not be given by equation (28). Hence, a necessary condition is that  $z_t$  mean-reverts at a rate  $\alpha$ .  $\square$

## Appendix 2: Properties of Lognormal *LIBOR* Rates (Proof of Lemma 2)

From Cox, Ingersoll and Ross (1981), the futures rate is equal to the expectation of the *LIBOR* under the equivalent martingale measure,  $F_{t,t+k} = E_t(r_{t+k})$ . Since by assumption  $r_{t+k}$  is lognormal, under the EMM, with a conditional logarithmic mean and annualised volatility of  $\mu(t, t+k, t+k)$  and  $\sigma(t, t+k, t+k)$ , we have

$$F_{t,t+k} = E_t(r_{t+k}) = \exp \left[ \mu(t, t+k, t+k) + \frac{kn}{2} \sigma^2(t, t+k, t+k) \right]$$

Now since

$$F(t, t+k) = E_t(r_{t+k})$$

the expectation of the futures rate is

$$E_0[F(t, t+k)] = E_0(r_{t+k}), \quad (32)$$

by the law of iterated expectations.

Taking the logarithm of equation (32) and using the relationship of the mean and variance of lognormal variables, we have

$$\mu(0, t, t+k) + \frac{tn}{2} \sigma^2(0, t, t+k) = \mu(0, t+k, t+k) + \frac{(t+k)n}{2} \sigma^2(0, t+k, t+k). \quad (33)$$

From the lognormality of  $r_{t+k}$ ,

$$(t+k)n\sigma^2(0, t+k, t+k) = \text{var}_0[\mu(t, t+k, t+k)] + kn\sigma^2(t, t+k, t+k). \quad (34)$$

Moreover,

$$F_{t,t+k} = \exp\left[\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)\right]$$

$$\text{var}_0[\mu(t, t+k, t+k)] = nt\sigma^2(0, t, t+k). \quad (35)$$

Substituting equations (35) into (34), and then (34) into (33), yields

$$\mu(0, t, t+k) = \mu(0, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k).$$

From the definition of the futures interest rate as the expectation of the future spot interest rate under the risk-neutral measure, it follows that:

$$\mu(t, t+k, t+k) = f_{t,t+k} - \frac{kn}{2}\sigma^2(t, t+k, t+k),$$

$$\mu(0, t+k, t+k) = f_{0,t+k} - \frac{kn}{2}\sigma^2(0, t+k, t+k),$$

□

### Appendix 3: The Volatilities and Correlations of Futures Rates (Proof of Proposition 3)

From Proposition 2, we have

$$\begin{aligned}\text{var}_{t-1}(f_{t,t+k}) &= a_k^2 \text{var}_{t-1}(f_{t,t}) + b_k^2 \text{var}_{t-1}(f_{t,t+1}) \\ &+ 2a_k b_k \text{cov}_{t-1}(f_{t,t}, f_{t,t+1})\end{aligned}\quad (36)$$

and

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+k}) = a_k \text{var}_{t-1}(f_{t,t}) + b_k \text{cov}_{t-1}(f_{t,t}, f_{t,t+1}). \quad (37)$$

First, we evaluate  $\text{var}_{t-1}(f_{t,t+1})$  and  $\text{cov}_{t-1}(f_{t,t}, f_{t,t+1})$ . The time  $t+1$  spot rate is

$$f_{t+1,t+1} = \mu(0, t+1, t+1) + [f_{t,t} - \mu(0, t, t)](1-c) + y_t + \epsilon_{t+1}.$$

Hence, the conditional expectation at time  $t$  is

$$\mu(t, t+1, t+1) = \mu(0, t+1, t+1) + [f_{t,t} - \mu(0, t, t)](1-c) + y_t.$$

But, from Lemma 1, we have

$$\mu(t, t+1, t+1) = f_{t,t+1} - \frac{n}{2} \sigma^2(t, t+1, t+1).$$

Hence, we can write

$$f_{t,t+1} = \mu(0, t+1, t+1) + \frac{n}{2} \sigma^2(t, t+1, t+1) + [f_{t,t} - \mu(0, t, t)](1-c) + y_t.$$

It then follows directly that

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) = (1-c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(y_t, f_{t,t}),$$

where  $\text{cov}_{t-1}(y_t, f_{t,t}) = \text{cov}_{t-1}(\nu_t, \epsilon_t)$ . Hence,

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+1}) = (1-c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(\nu_t, \epsilon_t). \quad (38)$$

Similarly

$$\text{var}_{t-1}(f_{t,t+1}) = (1-c)^2 \text{var}_{t-1}(f_{t,t}) + 2(1-c) \text{cov}_{t-1}(\nu_t, \epsilon_t) + \text{var}_{t-1}(\nu_t). \quad (39)$$

Substitution of (38) and (39) in (36) and (37) then yields

$$\begin{aligned}\text{var}_{t-1}(f_{t,t+k}) &= a_k^2 \text{var}_{t-1}(f_{t,t}) \\ &+ b_k^2 \left[ (1-c)^2 \text{var}_{t-1}(f_{t,t}) + 2(1-c) \text{cov}_{t-1}(\nu_t, \epsilon_t) + \text{var}_{t-1}(\nu_t) \right] \\ &+ 2a_k b_k [(1-c) \text{var}_{t-1}(f_{t,t}) + \text{cov}_{t-1}(\nu_t, \epsilon_t)]\end{aligned}$$

and

$$\text{cov}_{t-1}(f_{t,t}, f_{t,t+k}) = [(1-c)^k] \text{var}_{t-1}(f_{t,t}) + b_k \text{cov}_{t-1}(\nu_t, \epsilon_t).$$

Finally, using the annualised volatility definitions then yields:

$$\begin{aligned}\sigma^2(t-1, t, t+k) &= a_k^2 \sigma^2(t-1, t, t) + b_k^2 \sigma^2(t-1, t, t+1) \\ &+ 2a_k b_k [(1-c)^k \sigma^2(t-1, t, t) + 2\rho \sigma(t-1, t, t) \sigma_\pi]\end{aligned}$$

and the correlation of the spot and futures rates is therefore given by:

$$\rho(t-1, t, t+k) = \frac{(1-c)^k \sigma^2(t-1, t, t) + b_k \rho \sigma(t-1, t, t) \sigma_\pi}{\sigma(t-1, t, t) \sigma(t-1, t, k)}$$

where

$$\sigma^2(t-1, t, t+1) = (1-c)^2 \sigma^2(t-1, t, t) + \sigma_\pi^2 + 2(1-c) \rho \sigma_\pi \sigma(t-1, t, t)$$

and where  $\rho$  is the correlation between the errors  $\epsilon_t$  and  $\nu_t$ .  $\sigma_\pi$  is the volatility of  $\pi$ .  $\square$

## Appendix 4: Generalisation of the Model to Three Factors

In this appendix, we show that our model can be generalised to three-factors. We present the following proposition, based on three variables,  $x$ ,  $y$  and  $z$ :

**Proposition 4** (*General Cross-Sectional Relationship for the Change in Expected Values for the Three-Factor Case.*)

If the variable  $x_t$  follows the time series process

$$x_{t+1} = (1 - c)x_t + y_t + \epsilon_{t+1}$$

where

$$y_{t+1} = (1 - \alpha)y_t + z_t + \nu_{t+1}$$

and

$$z_{t+1} = (1 - \beta)z_t + \eta_{t+1}$$

the conditional expectation of  $x_{t+k}$  is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}) + c_k E_t(x_{t+2})$$

where

$$\begin{aligned} a_k &= a'_k - (1 - c)b'_k - (1 - c)(1 - \alpha)c'_k \\ b_k &= b'_k - (1 - c)c'_k - (1 - \alpha)c'_k \\ c_k &= c'_k \end{aligned}$$

where

$$\begin{aligned} a'_k &= (1 - c)^k \\ b'_k &= \sum_{\tau=1}^k (1 - c)^{k-\tau} (1 - \alpha)^{\tau-1} \\ c'_k &= \sum_{\tau=2}^k (1 - c)^{k-\tau} b''_{\tau-1} \end{aligned}$$

where

$$b_k'' = \sum_{\tau=1}^k (1-\alpha)^{k-\tau} (1-\beta)^{\tau-1}$$

*Proof.*

Successive substitution and taking expectations yields for the  $z_t$  variable:

$$E_t(z_{t+k}) = (1-\beta)^k z_t, \quad (40)$$

Similarly, for the  $y_t$  variable, using (40):

$$E_t(y_{t+k}) = (1-\alpha)^k y_t + \sum_{\tau=1}^k (1-\alpha)^{k-\tau} (1-\beta)^{\tau-1} z_t, \quad (41)$$

Successive substitution for  $x_t$ , using (41) then yields

$$\begin{aligned} E_t(x_{t+k}) &= (1-c)^k x_t + \sum_{\tau=1}^k (1-c)^{k-\tau} (1-\alpha)^{\tau-1} y_t \\ &+ \sum_{\tau=2}^k (1-c)^{k-\tau} \left[ \sum_{s=1}^{\tau-1} (1-\alpha)^{\tau-1-s} (1-\beta)^{s-1} \right] z_t, \quad (42) \end{aligned}$$

The proposition then follows by substitution of equations (40) and (41) in (42).

□

## References

- Backus, D. and S. Zin, 1994, "Reverse Engineering the Yield Curve," Working Paper, New York University.
- Balduzzi, P., S.R.Das, S. Foresi and R. Sundaram, 1996, "A Simple Approach to Three-Factor Models of Interest Rates," *Journal of Fixed Income* **6**.
- Balduzzi, P., S.R.Das, and S. Foresi, 1998, "The Central Tendency: A Second Factor in Bond Yields," *The Review of Economics and Statistics*, **80**, 62–72.
- Black, F., 1976, "The Pricing of Commodity Contracts," *Journal of Financial Economics*, **3**, 167–179.
- Black, F., E. Derman, and W. Toy, 1990, "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options," *Financial Analysts Journal*, **46**, 33–39.
- Black, F. and P. Karasinski, 1991, "Bond and Option Pricing when Short Rates are Lognormal," *Financial Analysts Journal*, **47**, 52–59.
- Bliss, R.R. and D.C. Smith, 1998, "The Elasticity of Interest Rate Volatility: Chan, Karolyi, Longstaff, and Sanders Revisited," (1998) Working Paper, Federal Reserve Bank of Atlanta.
- Brace, A., D. Gatarek, D. and M. Musiela, 1997, "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, **7**, 127–155.
- Brennan, M.J. and E.S. Schwartz, 1979, "A Continuous-Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, **3**, 135–155.
- Brenner R.J., R.H. Harges and K.F. Kroner, 1996, "Another Look at Models of the Short-Term Interest Rate," *Journal of Financial and Quantitative Analysis*, **31**, 85-107.
- Bühler, W., Uhrig-Homburg, M., Walter, U. and T. Weber, 1999, "An Empirical Comparison of Forward-Rate and Spot-Rate Models for Valuing Interest-Rate Options," *Journal of Finance*, **54**, 269–305.

- Chan, K.C., Karolyi, G.A., Longstaff, F.A. and A.B. Saunders, 1992, "An Empirical Comparison of Alternative Models of the Short-term Interest Rate," *Journal of Finance*, **47**, 1209–1227.
- Constantinides, G.M., 1992, "A Theory of the Nominal Term Structure of Interest Rates," *Review of Financial Studies*, **5**, 531–552.
- Cox, J.C., J.E. Ingersoll, and S.A. Ross, 1981, "The Relationship between Forward Prices and Futures Prices," *Journal of Financial Economics*, **9**, 321–46.
- Cox, J.C., J.E. Ingersoll, and S.A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, **53**, 385–407.
- Dai, Q. and Singleton, K.J., 2000, "Specification Analysis of Affine Term Structure," *Journal of Finance*, **55**, 1943–78.
- Duffie, D. and R. Kan, 1996, "A Yield -Factor Model of Interest Rates," *Mathematical Finance*, **6**, 379–406.
- Eom, Y. H. 1994, "In Defence of the Cox, Ingersoll and Ross Model: Some Empirical Evidence," Working paper, New York University.
- Gong, F. and E.M. Remolona, 1997, "Two Factors along the Yield Curve," *The Manchester School Supplement*, **No.65**, 1–31.
- Heath, D., R.A. Jarrow, and A. Morton, 1990a, "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation," *Journal of Financial and Quantitative Analysis*, **25**, 419–440.
- Heath, D., R.A. Jarrow, and A. Morton, 1990b, "Contingent Claim Valuation with a Random Evolution of Interest Rates," *Review of Futures Markets*, **9**, 55–75.
- Heath, D., R.A. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, **60**, 77–105.
- Ho, T.S.Y. and S.B. Lee, 1986, "Term Structure Movements and Pricing of Interest Rate Claims," *Journal of Finance*, **41**, 1011–1029.

- Ho, T.S., R.C. Stapleton, and M.G. Subrahmanyam, 1995, "Multivariate Binomial Approximations for Asset Prices with Non-Stationary Variance and Covariance Characteristics," *Review of Financial Studies*, **8**, 1125–1152.
- Hull, J. and A. White, 1990, "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, **3**, 573–92.
- Hull, J. and A. White, 1993, "One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities," *Journal of Financial and Quantitative Analysis*, 235–254.
- Hull, J. and A. White, 1994, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, **2**, 37–48.
- Jamshidian, F., 1989, "An Exact Bond Option Formula," *Journal of Finance*, **44**, 205–209.
- Jamshidian, F., 1991, "Bond and Option Evaluation in the Gaussian Interest Rate Model and Implementation," *Research in Finance*, **9**, 131–170.
- Jegadeesh, N. and G. G. Pennacchi, 1996, "The Behavior of Interest Rates Implied by the Term Structure of Eurodollar Futures," *Journal of Money Credit and Banking* **28**, 426–451.
- Lasdon, L.S., A.D. Waren, A.Jain and M. Ratner, "Design and Testing of a Generalised Reduced Gradient Code for Non-Linear Programming," *ACM Transactions on Mathematical Software*, 4(1), pp 34-50.
- Longstaff, F.A. and E.S. Schwartz, 1992, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance*, **47**, 1259–1282.
- Miltersen, K.R., K. Sandmann and D. Sondermann, 1997, "Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates," *Journal of Finance*, **1**, 409–430.
- Pliska, *Introduction to Mathematical Finance: Discrete Time Models*, Oxford: Blackwell, 1997.

- Sundaresan, S. 1991, "Futures Prices on Yields, Forward Prices, and Implied Forward Prices from Term Structure", *Journal of Financial and Quantitative Analysis*, **26**, 409-424.
- Turnbull, S.M. and F. Milne, 1990, "A Simple Approach to Interest-Rate Option Pricing," *Review of Financial Studies*, **4**, 87-120.
- Stapleton, R.C. and M.G. Subrahmanyam, 1993, "Analysis and Valuation of Interest Rate Options," *Journal of Banking and Finance*, **17**, 1079-1095.
- Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, **5**, 177-188.

**Table 1**  
**Notation for the Mean and Volatility of Spot and Futures Rates**

Time Period	(1) 0	(2) $t$	(3) $T$
Spot prices and interest rates for $m$ -year money	$\mu(0, t, t)$ Unconditional logarithmic mean of $i_t$	$P_t$ Zero bond price at $t$ for delivery of \$1 at $(t + m)$	$P_T$ Zero bond price at time $T$ for delivery of \$1 at time $T + m$
	$\sigma(0, t, t)$ Unconditional (annualised) volatility of $i_t$	$i_t = F_{t,t}$ $m$ -year interest rate at time $t$	$i_T = F_{T,T}$ $m$ -year interest rate at time $T$
Futures interest rates for bonds maturing at time $T + m$	$\mu(0, t, T)$ Mean of $f_{t,T}$ $\sigma(0, t, T)$ Unconditional (annualised) volatility of $F_{t,T}$	$F_{t,T}$ futures interest rate at $t$ for delivery at $T$ ( $m$ -year money) $f_{t,T}$ Logarithm of $F_{t,T}$ $\mu(t, T, T)$ Conditional mean of $f_{T,T}$ $\sigma(t, T, T)$ Conditional (annualised) volatility of $F_{T,T}$	

**Table 2: Historical Volatility and Correlation Estimates for Eurodollar Futures Rates, 1995-1999**

Mos.	vol. (%)	Spot Rate	Fut. 3	Fut. 6	Fut. 9	Fut. 12	Fut. 15	Fut. 18	Fut. 21	Fut. 24	Fut. 27	Fut. 30	Fut. 33	Fut. 36	Fut. 39	Fut. 42	Fut. 45	Fut. 48	Fut. 51	Fut. 54	Fut. 57
Spot	08.20																				
3	13.42	0.63																			
6	17.55	0.56	0.97																		
9	19.89	0.49	0.95	0.98																	
12	20.18	0.52	0.94	0.98	0.98																
15	20.57	0.50	0.96	0.98	0.98	0.99															
18	19.76	0.48	0.94	0.98	0.97	0.98	0.99														
21	18.86	0.43	0.93	0.96	0.97	0.97	0.99	0.99													
24	17.70	0.46	0.92	0.96	0.96	0.98	0.99	0.99	0.99												
27	17.36	0.44	0.92	0.94	0.94	0.95	0.98	0.99	0.99	0.99											
30	16.72	0.43	0.91	0.94	0.93	0.94	0.97	0.99	0.99	0.99	1.00										
33	16.25	0.38	0.89	0.92	0.92	0.92	0.96	0.97	0.99	0.98	0.99	1.00									
36	15.46	0.41	0.89	0.91	0.91	0.93	0.96	0.97	0.98	0.99	0.99	1.00	1.00								
39	15.35	0.38	0.88	0.89	0.89	0.90	0.95	0.95	0.97	0.97	0.99	0.99	0.99	0.99							
42	15.15	0.37	0.87	0.88	0.88	0.89	0.93	0.95	0.96	0.96	0.98	0.99	0.99	0.99	1.00						
45	14.96	0.33	0.84	0.85	0.85	0.86	0.91	0.93	0.95	0.94	0.97	0.97	0.99	0.98	0.99	0.99					
48	14.44	0.35	0.85	0.86	0.87	0.86	0.92	0.93	0.95	0.95	0.97	0.98	0.99	0.98	0.99	1.00	1.00				
51	14.30	0.32	0.83	0.84	0.84	0.85	0.91	0.92	0.94	0.94	0.96	0.97	0.98	0.98	0.99	0.99	1.00	1.00			
54	14.02	0.32	0.81	0.82	0.82	0.83	0.89	0.91	0.93	0.93	0.95	0.96	0.98	0.97	0.99	0.99	1.00	0.99	1.00		
57	13.92	0.28	0.80	0.81	0.82	0.82	0.88	0.90	0.92	0.92	0.95	0.96	0.97	0.97	0.98	0.99	0.99	0.99	1.00	1.00	
60	13.23	0.30	0.80	0.81	0.81	0.83	0.88	0.90	0.92	0.93	0.95	0.96	0.97	0.97	0.98	0.99	0.99	0.99	1.00	0.99	0.9

The table presents estimates of the annualised volatilities of, and correlation coefficients between, the three-month spot *LIBOR* and Eurodollar futures rates for the period January 1, 1995 to December 31, 1999, using quarterly data for all contracts with maturities from 3 months to 60 months. Since the futures maturity dates are fixed, we interpolate between two adjacent contracts to construct a time-series of futures prices and rates for fixed maturity periods. The first column provides the maturity of the futures contract, and the second column the annualised volatility, i.e, the annualised standard deviation of the logarithm of the ratios of the futures rates, for each maturity. The rest of the table shows the coefficients of correlation between the futures rates of various maturities.

**Table 3: Parameter Values of the Two-Factor Model Using Historical Volatility and Correlation Estimates**

	$\sigma_r$	$\sigma_\pi$	$c$	$\alpha$	$\rho$	$RMSE_\sigma$	$RMSE_\rho$	$RMSE$
<b>Panel A</b>								
Constrained Minimization with $\rho = 0$								
Fit to Volatilities	0.082	0.112	0.028	0.552	0	0.028*	0.175	0.125
Fit to Volatilities and Correlations	0.093	0.087	0.034	0.407	0	0.058	0.144	0.110*
<b>Panel B</b>								
Unconstrained Minimization								
Fit to Volatilities								
Solution 1	0.082	0.135	0.624	0.028	-0.167	0.026	1.353	0.957
Solution 2	0.082	0.151	0.028	0.624	-0.473	0.026*	1.353	0.957
Fit to Volatilities and Correlations								
Solution 1	0.087	0.084	0.040	0.370	0.057	0.062	0.140	0.108*
Solution 2	0.087	0.090	0.370	0.040	0.370	0.062	0.140	0.108

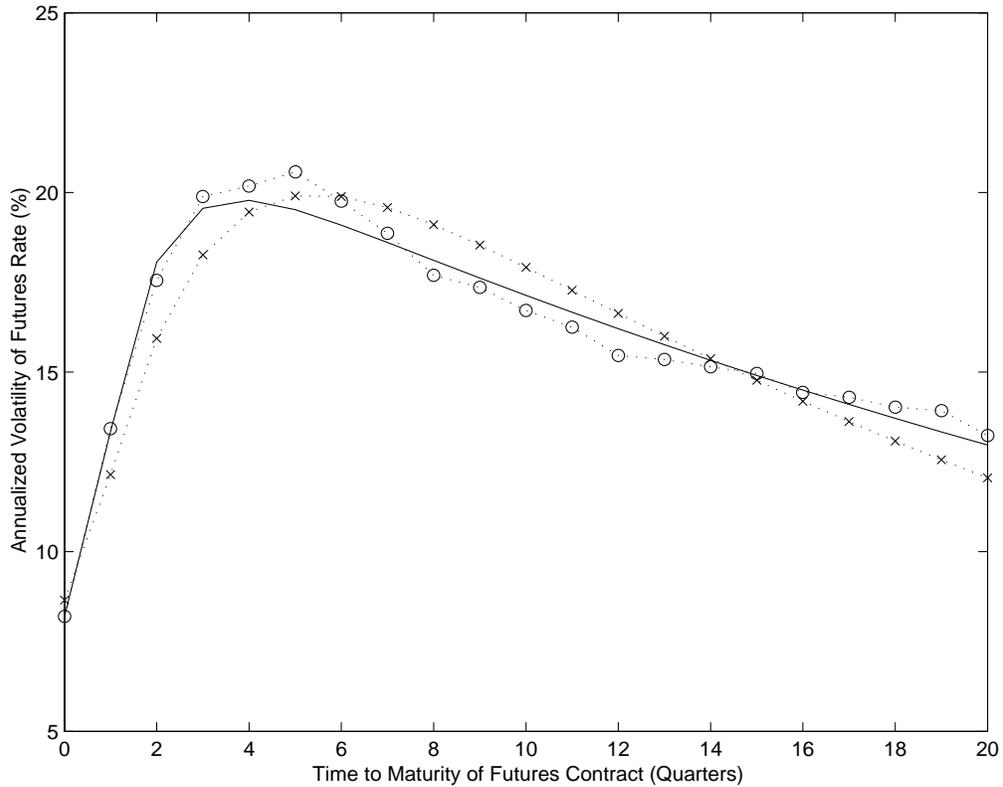
Each panel of the table presents the parameter estimates of the two-factor model using two different methods of calibration, one by fitting the model to the historical volatilities and other by fitting it to both historical volatilities and correlation coefficients. The parameters estimated are the volatility of the short term interest rate,  $\sigma_r$ , the volatility of the futures premium,  $\sigma_\pi$ , the mean-reversion coefficient,  $c$ , the measure of persistence,  $\alpha$ , and the correlation between the errors,  $\rho$ . The historical volatilities and correlation coefficients are based on quarterly data for the three-month spot *LIBOR* and Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, for all contracts with maturities from 3 months to 60 months. The parameters are estimated by minimising the root mean squared error (RMSE) using the algorithm of Lasdon et.al. (1978). The RMSE with \* are minimised values.

**Table 4: Sensitivity of Parameter Values and RMSE to Changes in the Correlation between the Errors**

$\rho =$	-0.3	-0.1	0	0.1	0.3	Optimal 0.37	$\rho$ 0.06
$\sigma_r$	0.120	0.103	0.093	0.082	0.084	0.087	0.087
$\sigma_\pi$	0.094	0.090	0.087	0.081	0.063	0.090	0.084
$c$	0.01	0.02	0.03	0.05	0.15	0.37	0.04
$\alpha$	0.54	0.46	0.41	0.34	0.10	0.04	0.37
RMSE	0.148	0.118	0.109	0.109	0.124	0.108	0.108

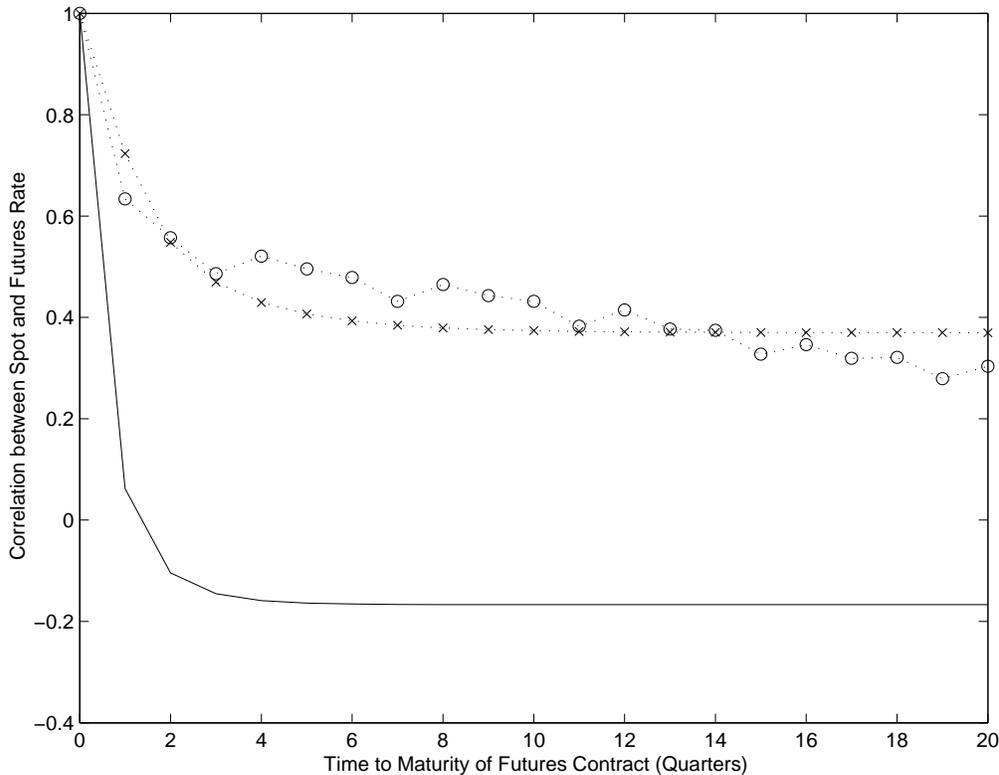
The table presents the parameter estimates of the two-factor model calibrating to both the historical volatilities and the correlation coefficients. The parameters estimated are the volatility of the short term interest rate,  $\sigma_r$ , the volatility of the futures premium,  $\sigma_\pi$ , the mean-reversion coefficient,  $c$ , the measure of persistence,  $\alpha$ , and the correlation between the errors,  $\rho$ . The historical volatilities and correlation coefficients are based on quarterly data for the three-month spot *LIBOR* and Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, for all the contracts with maturities from 3 months to 60 months. The parameters are estimated by minimising the root mean squared error (RMSE) using the algorithm of Lasdon et. al. (1978). This table shows the different parameter values and the RMSE when the correlation coefficient between the errors,  $\rho$ , is fixed at different levels. The lowest RMSE arises when the correlation coefficient is 0.37 or 0.06.

Figure 1: The Term Structure of Volatility of Futures Rates



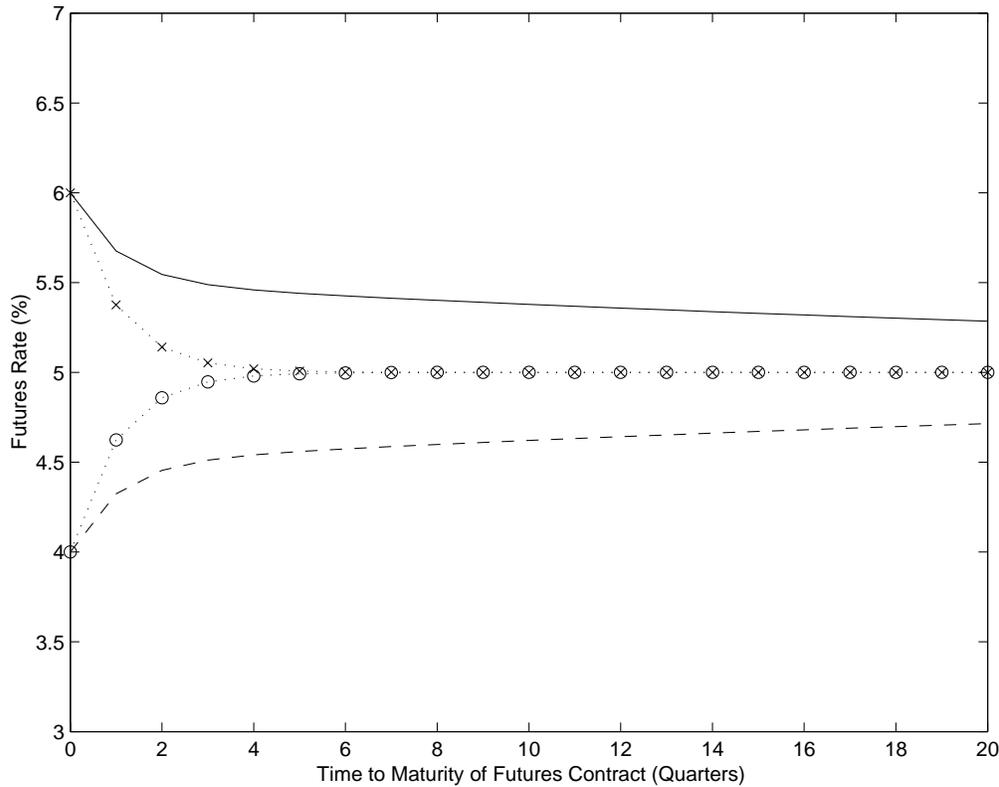
The figure shows the term structure of annualised volatilities of futures rates from the two-factor model for two alternative calibrations and the actual historical volatilities, based on Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, using quarterly data for all contracts with maturities from 3 months to 60 months. The solid line represents the calibration using only the historical volatilities (mean reversion,  $c = 0.62$ , persistence,  $\alpha = 0.03$ ), while the line with crosses represents the calibration using both historical volatilities and correlation coefficients (mean reversion,  $c = 0.04$ , persistence,  $\alpha = 0.37$ ), both for the unconstrained case without restrictions on the correlations between the errors. The line with circles represents the graph for the actual historical volatilities.

Figure 2: The Term Structure of Correlation Coefficients between the Spot and Futures Rates



The figure shows the term structures of correlations between the spot rate and the  $k^{\text{th}}$  futures rate for the two-factor model, for two alternative calibrations and the actual historical correlation coefficients, based on Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, using quarterly data for all contracts with maturities from 3 months to 60 months. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (24). The solid line represents the calibration using only the historical volatilities (mean reversion,  $c = 0.62$ , persistence,  $\alpha = 0.03$ ), while the line with crosses represents the calibration using both historical volatilities and correlation coefficients (mean reversion,  $c = 0.04$ , persistence,  $\alpha = 0.37$ ), both for the unconstrained case without restrictions on the correlations between the errors. The line with circles represents the graph for the actual historical volatilities.

Figure 3: The Term Structure of Futures Rates



The figure illustrates the term structures of futures rates for the two factor model calibrated to both historical volatilities and correlation coefficients, based on Eurodollar futures prices for the period January 1, 1995 to December 31, 1999, using quarterly data for all contracts with maturities from 3 months to 60 months (mean reversion,  $c = 0.04$ , persistence,  $\alpha = 0.37$ ). The base case (not shown) is a flat term structure, where the rates are 5% for all maturities. In the one-factor case, the short rate shifts up or down by 1%. The shift up is represented by the line with crosses, while the shift down is represented by the line with circles. In the two-factor case, the premium factor also moves up or down by 0.3% from the base case, in the same direction. The shift down is similar in logarithmic terms. The solid lines represent the term structures for the two-factor model, for the up and down cases.