The Libor Market Model: A Recombining Binomial Tree Methodology

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April 19, 2005

Abstract

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We propose an implementation of the Libor Market Model, adapting the recombining node methodology of Ho, Stapleton and Subrahmanyam (1995). Initial tests on one-factor and two-factor versions of the model suggest that the method provides a fast and accurate approach for the valuation of path dependent interest rate derivatives such as Bermudan-style swaptions. The lattice based approach illustrated here provides an efficient alternative to Monte-Carlo simulation implementation of the Libor Market Model.

1 Introduction

The Libor Market Model (LMM) is the most common implementation in practice of the general Heath, Jarrow and Morton (1990) forward rate approach to the valuation of interest-rate derivatives. First proposed by Miltersen, Sandmann and Sondermann (1997) (MSS) and Brace, Gatarek and Musiella (1997) (BGM), the model assumes that the London Interbank Offer Rate (Libor) has a conditional probability distribution which is lognormal. This paper addresses two problems that arise with the LMM. First, the proof of the continuous-time LMM is somewhat obscure. Hull (2003), for example simply states the drift of the forward rates, under the risk-neutral measure, without proof. However, without a knowledge of why the drift is as stated in the model, it is hard to use the model with confidence. We show here that the drift of the forward rate can be derived very simply from the pricing of Forward Rate Agreements (FRAs) in a no-arbitrage setting.

Second, multi-factor versions of the LMM are difficult to implement, especially for the pricing of Bermudan-style swaptions. We employ the recombining binomial-tree methodology of Ho, Stapleton and Subrahmanyam (1995) (HSS) to construct a swaption pricing model, which does not have to rely on Monte-Carlo simulation or the lower-bound approximations commonly employed[1] As an illustration, we implement a two-factor example of the LMM and price European-style and fixed tail Bermudan-style swaptions.

The standard Monte-Carlo simulation implementation of the LMM leads to an exploding tree of forward and spot rates. This is caused by the stochastic drift of the forward rates (see for example Hull, p.577-80). In our implementation the drift is captured by using conditional probabilities (of up-moves in the processes) which are time and state dependent. In the LMM, the drift depends on the forward rate, but since in our methodology the forward rates do not explode, then neither do the number of conditional probabilities. The main advantage of such lattice models such as ours is that path-dependent Bermudan-style options can be valued using an optimal exercise strategy, given the interest-rate process. The main disadvantage is that the process itself is an approximation to the true process, where the degree of approximation depends on the fineness of the lattice structure.

The outline of this article is as follows. In section 2, we define the Black model, as it is applied to the pricing of interest-rate options such as caps/floors and swaptions. We also derive some preliminary results regarding the drift of forward bond prices under the risk-neutral measure[2]. In section 3, we derive a version of the discrete-time LMM, following

[1]For a recent attempt to survey developments in the application of Monte-Carlo techniques to the pricing of Bermudan swaptions see Piterbarg (2004/5).

[2]By risk-neutral measure we mean the measure under which futures prices follow a martingale, where the

the development in Poon and Stapleton (2005). In this discrete-time model, the drift (i.e. the change in the expected value) of the T-maturity forward rate can be determined by pricing a T-maturity forward rate agreement (FRA) and applying an extension of Stein's lemma adapted for lognormal variables[3]. In section 4, we detail how the HSS-Nelson and Ramaswarmy (1990) methodology can be applied in the case of the multi-factor LMM by fixing the conditional probabilities in the forward interest-rate process. In section 5, we apply the model to pricing of bonds, caps and swaptions. Section 6, discusses the calibration of the model, to cap volatilities and to European-style swaption volatility quotes. In section 7, we report on the performance of the model measured by its ability to reproduce the Black-model cap prices and swaption prices from Monte-Carlo implementation, as reported in Andersen (2000).

2 Main Features of Libor Market Models

The LMM is a term-structure model which recovers caplet and floorlet values which are consistent with the market practice of applying the Black model to price options on interest rates, defined on a Libor basis. Many market participants build LMMs and use them to price path-dependent interest-rate derivatives such as Bermudan-style swaptions. The main assumption of the model is that the forward rate is conditional lognormal under the risk-neutral measure. Since in the model the drift of the forward rate is stochastic, the Libor is not unconditional log-normal. If it was, then as in the Black and Karasinski (1991) spot-rate model, the Black model would not hold for caplets and floorlets, due to the effects of stochastic discounting[4].

The Black model for a caplet is given by:

Definition 1 [The Black Model: Interest-Rate Caplet]

The price of a caplet with maturity T at time t, if the Black model holds is

$$caplet_{t,T} = \frac{A}{1 + f_{t,t+T}\delta}\delta[f_{t,t+T}N(d_1) - kN(d_2)]B_{t,t+T} \tag{1}$$

period length is the reset period for the Libor loans. Typically, this period length will be three or six months. The theoretical risk-neutral measure often used in the literature is based on a period of infintessimal length.

[3] See Stein (1973)'s lemma for multivariate normal variables

[4] Pricing a caplet (floorlet) using the Black-Karasinski model leads to caplet (floorlet) prices which are less than (greater than) those obtained from the Black model.

where

$$d_1 = \frac{ln(\frac{f_{t,t+T}}{k}) + \sigma(T)^2 T/2}{\sigma(T)\sqrt{T}}, \;\; d_2 = d_1 - \sigma(T)\sqrt{T}$$

where

A is the principal value of the caplet.

$B_{t,t+T}$ is the value at t of a zero-coupon bond paying 1 unit of currency at $t+T$.

δ is the interest-rate reset interval (ex. 3 months) as a proportion of a year.

k is the strike rate of the caplet.

$f_{t,t+T}$ is T-period forward Libor at time t.

$\sigma(T)$ is the volatility of T-period Libor.

If this pricing equation holds for all T and for all strike rates, k, we say that the Black model holds for caplets [5] .

The Libor Market Model is a model of the stochastic evolution of interest rates that is consistent with the above formula holding for all caplets, with maturities $T = 1, 2, ..., N$ and all strike rates k. The model allows other interest-rate dependent contingent claims, such as European-style and Bermudan-style swaptions, to be priced in a way that is consistent with the pricing of the caplets.

The derivation of the LMM uses many of the concepts that are standard in financial theory. In particular, it uses the ideas of the risk-neutral measure, forward parity, and no-arbitrage asset pricing relationships. These ideas are well documented in texts such as Pliska (1997) and Poon and Stapleton (2005). For convenience, we re-state the most important results here. Since we will be concerned with the pricing of zero-coupon bonds, the relevant ideas concern zero-dividend paying assets. We have the following results.

For a zero-dividend paying asset:

[5] However, caplets are often quoted in the market by solving equation (1) for $\sigma(T,k)$, for different forward maturities T and strike prices k. In this case it is being used merely as a quotation system.

1. The no-arbitrage spot asset price is given by[6]

$$S_t = B_{t,t+1}E_t[B_{t+1,t+2}E_{t+1}[B_{t+2,t+3}[....E_{t+T-1}[B_{t+T-1,t+T}S_T]]]],$$

   where the expectation is taken under the period-by-period risk-neutral measure.

2. The T-period forward price of the asset is given by

$$F_{t,t+T} = S_t/B_{t,t+T}$$

3. The expected one-period-ahead forward price of the asset is[7]

$$E_t(F_{t+1,t+T}) = F_{t,t+T} - cov_t(F_{t+1,t+T}, B_{t+1,t+T})\frac{B_{t,t+1}}{B_{t,t+T}}.$$

Since forward rates are closely related to forward prices of zero-coupon bonds, and since we will be interested in the drift of forward rates, we now apply these results to price forward contracts on zero-coupon bonds. In the case of zero-coupon bonds we have the following result:

Lemma 1 (Poon and Stapleton (2005), ch 7) *When expectations are taken under the risk-neutral measure:*

1. *The drift of the T-period forward price of a one-period maturity zero-coupon bond is*

$$E_t(B_{t+1,t+T,t+T+1}) - B_{t,t+T,t+T+1} = \frac{B_{t,t+1}}{B_{t,t+T}}cov_t(B_{t+1,t+T,t+T+1}, B_{t+1,t+T})$$

2. *The one-period ahead forward price of a long maturity bond is:*

$$B_{t,t+1,t+T} = E_t(B_{t+1,t+2}B_{t+1,t+2,t+T}),$$

where $B_{t,t+\tau,t+T}$ is the τ period forward price of a bond with maturity date $t+T$.

Lemma 1 would be directly useful if we were building a stochastic process of forward bond prices. It shows how the drift of the forward price depends upon the covarariance of the forward prices. However, the LMM is a model of forward *rates*. In this case, a similar effect of covariances determines the drift of forward rates. In the following section, we use a corollary of the above lemma in the analysis of the forward rate drift.

[6]See, for example Pliska (1997), chapter 2

[7]This follows from taking expectations and using the definition of covariance.

3 The Libor Market Model

In this section we derive a discrete-time version of the Brace, Gatarek and Musiela (1997) LMM. The analysis closely follows that in Poon and Stapleton (2005), chapter 7. The Libor Market Model is constructed by forming a process for the evolution of the forward rates for all maturities up to a given teminal date. Perhaps the most important element of the LMM is the drift of the forward Libor over any period. Here we show that the drift can be determined from the pricing of forward rate agreements (FRAs). By definition FRAs have a zero value when issued and this valuation can be used to determine the drift of the forward rate.

We begin the derivation of the LMM by defining the following standard contract: A *Forward Rate Agreement* (FRA) on Libor, with maturity T, has a payoff

$$\frac{(f_{t+T,t+T} - k)\delta}{1 + f_{t+T,t+T}\delta}, \quad \text{at date } t + T,$$

where $f_{t+T,t+T}$ is the spot Libor at time $t + T$. Note that the definition assumes that the contract is settled at time $t + T$ on a discounted basis at time t. Here we assume, for simplicity, that δ is a constant. In practice, the precise payoff depends on the day count. Hence, the above contract can be thought of as a theoretical, or idealised FRA. We now have the following result:

FRA Pricing and the Drift of the Forward rate

We begin by stating the following corollary of Lemma 1: From *forward parity*, the one-period ahead forward price of a one-period bond is

$$B_{t,t+1,t+2} = E_t(B_{t+1,t+2}),$$

or, in terms of forward rates

$$E_t\left(\frac{1}{1 + f_{t+1,t+1}\delta}\right) = \frac{1}{1 + f_{t,t+1}\delta}.$$

Now we use this relationship to price a one-period FRA. Since the FRA must have a zero value when struck at the forward rate:

$$E_t\left[\frac{(f_{t+1,t+1} - f_{t,t+1})\,\delta}{1 + f_{t+1,t+1}\delta}\right] = 0,$$

where expectations are taken under the risk-neutral measure. It then follows that

$$E_t\left(\frac{f_{t+1,t+1}}{1+f_{t+1,t+1}\delta}\right)=\frac{f_{t,t+1}}{1+f_{t,t+1}\delta}.$$

Further, using the definition of covariance we have

$$E_t(f_{t+1,t+1})-f_{t,t+1}=-cov\left(f_{t+1,t+1},\frac{1}{1+f_{t+1,t+1}\delta}\right)(1+f_{t,t+1}\delta).\geq 0$$

The difference between the expected forward rate at $t+1$ and the forward rate at t, i.e. the drift of the forward rate over the first period, depends on the covariance of the rate with the bond price. In the case of the two period forward rate we have a similar, but more complicated result, which we state in the following result.

For small changes, $\frac{dx}{x}=d\ln x+k$, and

$$cov\left(\cdots\right)=cov\left[\ln f_{t+1,t+2},\ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\frac{1}{1+f_{t+1,t+1}\delta}\right)\right]\frac{f_{t,t+2}}{\left(1+f_{t,t+1}\delta\right)\left(1+f_{t,t+2}\delta\right)}$$

Hence

$$\begin{aligned}
& E_t\left[f_{t+1,t+2}\right]-f_{t,t+2} \\
= \; & -cov\left[\ln f_{t+1,t+2},\ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\frac{1}{1+f_{t+1,t+1}\delta}\right)\right]f_{t,t+2} \\
= \; & -cov\left[\ln f_{t+1,t+2},\ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\right)\right]f_{t,t+2}-cov\left[\ln f_{t+1,t+2},\ln\left(\frac{1}{1+f_{t+1,t+1}\delta}\right)\right]f_{t,t+2}
\end{aligned}$$

This shows that the drift of the two period ahead forward rate depends on the two covariance terms. We now evaluate these covariance terms using a well known property of normally distributed variables. Applying an extension of Stein's Lemma for lognormal variables, we

Hence,

$$\begin{aligned}
cov\left[\ln f_{t+1,t+T},\ln\left(\frac{1}{1+f_{t+1,t+\tau}\delta}\right)\right] &= E\left(\frac{-f_{t+1,t+\tau}\delta}{1+f_{t+1,t+\tau}\delta}\right)cov(\ln f_{t+1,t+T},\ln f_{t+1,t+\tau}) \\
&= \left(\frac{-f_{t,t+\tau}\delta}{1+f_{t,t+\tau}\delta}\right)\sigma_{T,\tau},
\end{aligned}$$

where, assuming now that covariances are independent of time t,

$$cov(\ln f_{t+1,t+1+T}, \ln f_{t+1,t+1+\tau}) \equiv \sigma_{T-1,\tau-1}.$$

Stein's lemma, applied above to the case of lognormal variables, provides the key to evaluating the covariance terms that determine the drift of the forward rates. It turns covariances in to logarithmic covariances. We now state the drift terms, in the general case, assuming first for simplicity that $\delta = 1$. We have, given Stein's Lemma, for small changes in forward rates

$$f_{t,t+2} = E_t\left[f_{t+1,t+2}\right] - f_{t,t+2}\frac{f_{t,t+2}}{1+f_{t,t+2}\delta}\sigma_{1,1} - f_{t,t+2}\frac{f_{t,t+1}}{1+f_{t,t+1}\delta}\sigma_{0,1}$$

which states that the drift is dependent, as in HJM, on a series of discounted covariances. We are now in a position to state the drift in the BGM version of the LMM. First we make an additional assumption. We assume that the covariances (of the logarithms of the forward rates) depend only on the maturity of the forward rates, i.e. they are not time dependent. We then can establish:

Proposition 1 *The BGM Model*

Given intertemporal stability of the covariances, if the period length t *to* $t+1$ *is also* δ *years, then the drift of the forward Libor is*

$$E\left[f_{t+1,t+T}\right] - f_{t,t+T} = f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1} + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1} + \cdots \quad (2)$$

In the LMM, the drift of the forward rate at a point in time depends upon the level of the rate. It also depends on the sum of a series of discounted covariances. Since, the drift depends on the forward rate, it is stochastic. This is the property that causes the problem in implementing the model. The stochastic drift can be handled by a Monte-Carlo simulation. But, to avoid this brute force approach we show in the following section that we can model the stochastic drift using adjusted conditional probabilities.

4 The HSS Recombining Node Methodology and the Libor Market Model

Ho, Stapleton and Subrahmanyam (1995) [HSS] suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process. An adaptation of this methodology has been used by Peterson, Stapleton and Subrahmanyam (2003) [PSS] to build a two-factor spot rate model of the term-structure. In this section we show how a similar application can be made in the case of the LMM. There are some differences in this case however. First, we will assume in this version of the LMM that the stochastic factors driving the term structure are independent log-Brownian motions. Hence, there is no mean reversion or correlation in the factors. The covariances between forward rates are generated by factor loadings on the factors.

We assume a given term structure of forward Libors, $f_{0,T},\ T = 0, 1, ..., N$, where N is the terminal date of the model and a corresponding set of caplet volatilities, $\sigma(T)\ T = 1, ..., N$. From these caplet volatilities we derive a set on N forward volatilities, using the bootstrap method, assuming that the forward rate volatilities depend on the forward maturity, T, and not on time t. We denote these as σ_T. As in Hull and White (2000), we now assume these volatilities are generated by a two-factor model, with factor loadings: $\beta_{1,T}$, $\beta_{2,T}$, where

$$\beta_{i,T} = \alpha_{i,T}\sigma_{T+1},\ T = 0, 1, ..., N - 1.$$

In the two-factor model, the exogenous factors, $\alpha_{i,T}$ are restricted by the relation

$$\alpha_{1,T}^2 + \alpha_{2,T}^2 = 1,$$

and $0 < \alpha_{1,T} < 1$. The covariance between any two forward rates is then given by

$$\sigma_{\tau,T} = \beta_{1,\tau}\beta_{1,T} + \beta_{2,\tau}\beta_{2,T}. \tag{3}$$

For convenience, we denote the part of this covariance generated by factor i as $\sigma_{\tau,T}(i) = \beta_{i,\tau}\beta_{i,T}$.

In HSS, a binomial process is used to approximate a stock-price process with a given volatility and drift structure. Here we use a similar methodology, applying it to each of the factors which generate the forward rates. We first split the drift of the forward rate into two parts: a drift which is due to the volatility of the first factor and a drift which is due to the volatility of the second factor. Since $\sigma_{\tau,T} = \sigma_{\tau,T}(1) + \sigma_{\tau,T}(2)$, the drift in equation (2) can be written

$$E\left[f_{t+1,t+T}\right] - f_{t,t+T} = f_{t,t+T}\frac{\delta f_{t,t+T}}{1 + \delta f_{t,t+T}}[\sigma_{T-1,T-1}(1) + \sigma_{T-1,T-1}(2)]$$

$$
\begin{aligned}
&+ \quad f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}[\sigma_{T-2,T-1}(1)+\sigma_{T-2,T-1}(2)] \\
&+ \quad \cdots
\end{aligned}
$$

and therefore

$$
\begin{aligned}
E\left[f_{t+1,t+T}\right]-f_{t,t+T} &= f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(1)+f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(1)+\cdots \\
&+ f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(2)+f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(2)+\cdots
\end{aligned}
$$

A straightforward implementation of the LMM would build a non-recombining binomial tree for each factor, using the factor loadings $\beta_{i,T}$ and the drifts above. The resulting bivariate tree would have an exploding number of nodes, but could be used to value interest-rate options using Monte-Carlo analysis. As an alternative, we build a recombining binomial tree for each factor using the techniques in HSS and Nelson and Ramaswamy (1990) and then capture the required drift by using state dependent conditional probabilities.

First, we denote the proportionate up and down movements in the log-binomial process due to factor $i=1,2$ as $u(T)(i)$ and $d(T)(i)$ respectively, for the forward rate with maturity T, where the up and down moves of the processes depend only on the maturity of the forward, T, and not on the time t. The T-period forward rate at time t, in state r, s, [after r down-moves in factor 1 and s down-moves in factor 2] is given by

$$
f_{t,t+T,r,s}=f_{0,t+T}[u_T(1)]^{t-r}[d_T(1)]^r[u_T(2)]^{t-s}[d_T(2)]^s \tag{4}
$$

where

$$
\begin{aligned}
d_T(i) &= \frac{2}{1+e^{2\beta_{i,T}\sqrt{\delta}}} \\
u_T(i) &= 2-d_T(i),
\end{aligned}
$$

for

$$
\begin{aligned}
t &= 1,2,...,N \\
T &= 0,1,...,N-t.
\end{aligned}
$$

Choosing the proportionate up and down moves, u and d, in this way ensures that the annualised volatility of the Tth forward is exactly σ_T, in the case where all probabilities in the tree are[8] 0.5. Also the tree of forward rates is recombining with $(t+1)^2$ nodes at time t.

[8] If all the conditional probabilities are 0.5, the binomial process has a volatility exactly equal to σ_T. However, when corrected to account for the drift, the probabilities will diverge from 0.5 and the volatility of the binomial process will understate that of the true process.

In Figure 1 we illustrate the recombining tree for the two-factor case. In this example, the volatility of the first factor declines over time, while the second factor has constant volatility. Note that, as in HSS, the tree is forced to recombine, in spite of the fact that the volatility of the first factor declines over time. The transition probabilities depend both on volatility structure and the drifts of the factors. For example, the conditional probability at time 1 of the two-period forward moving up at node (0,0), due to factor 1, is denoted as $q_{1,2,0,0}(1)$. In general for factor i, at time t in state (r, s) this is denoted as $q_{t,t+T,r,s}(i)$.

In order to fix the conditional probabilities in the binomial process, using the HSS methodology, we need to determine the drift of the logarithm of the forward rates. In HSS Theorem 1, it is shown that the binomial process converges to the required lognormal process if the conditional probabilities are chosen using the logarithmic regression of prices on previous prices. Here we follow the same logic, based on the logarithm of forward rates. Dividing the drift equation by $f_{t,t+T}$,

$$\begin{aligned} \frac{E\left[f_{t+1,t+T}\right] - f_{t,t+T}}{f_{t,t+T}} &= \frac{E\left[\Delta f_{t,t+T}\right]}{f_{t,t+T}} \\ &= \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1} + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1} + \cdots \end{aligned}$$

Then, for small changes, using Ito's lemma

$$\begin{aligned} \frac{E\left[d\ln f_{t,t+T}\right]}{f_{t,t+T}} &= \frac{E\left[df_{t,t+T}\right]}{f_{t,t+T}} - \frac{\sigma_{T,T}}{2} \\ &= \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1} + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1} + \cdots - \frac{\sigma_{T,T}}{2}, \end{aligned} \tag{5}$$

where $\sigma_{T,T} = \sigma_T^2$ is the variance of the Tth forward rate.

In the HSS binomial tree, the conditional probability of an up move in the T-maturity forward due to factor i, at time t at node r, s is

$$\begin{aligned} q_{t,t+T,r,s}(i) &= [m_{t,t+T,r,s}(i) + (t-r)\ln u_{T+1}(i) + r\ln d_{T+1}(i) - (t-r)\ln u_T(i) - r\ln d_T(i) \\ &- \ln d_T(i)]/[\ln u_T(i) - \ln d_T(i)], \end{aligned} \tag{6}$$

where $m_{t,t+T,r,s}(i)$ is the annualised logarithmic drift of factor i, and δ is the length of the period t to $t+1$. The drift of the forward rate in equation (5) can now be allocated between the two factors, choosing

$$m_{t,t+T,r,s}(i) = \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(i) + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(i) + \cdots - \frac{\sigma_{T,T}(i)}{2},$$

where $\sigma_{T,T}(1) + \sigma_{T,T}(2) = \sigma_{T,T}$.

This completes the description of the HSS application for the case where there is just one up or down move, for each factor, over each period of length δ. However, the general HSS methodology allows for an increase in the binomial density, n, (the number of up or down moves per period). When $n \neq 1$ the above formula generalises as in HSS. Also, as n increases, the variance of the forward rate process converges to the given volatilities. This follows from HSS, Theorem 1.

5 The Pricing of Derivatives in the LMM

In this section we describe the way various interest-rate derivatives are priced using the LMM. The important derivative products are interest-rate caps and floors, and European-style and Bermudan-style swaptions.

5.1 The Price of Caplets in the LMM

In this section we present a first test of the HSS procedure described above. Since the LMM is a process for interest rates (spot and forward) which is consistent with the Black model pricing of caplets, a good first test of our HSS approximation process for the forward rates is to examine how close are the caplet prices to the Black-model prices.

The payoff at time τ, on a European-style caplet with maturity τ, at node (r, s) (after r down-moves of the process for factor 1 and s down-moves of the process for factor 2) is defined as

$$cap_{\tau,\tau,r,s} = max[f_{\tau,\tau,r,s} - k, 0]\delta \frac{1}{1 + f_{\tau,\tau,r,s}\delta}.$$

We price the caplet by discounting back through the tree of states, using the appropriate conditional probabilities to compute the expected payoff on the caplet one period hence. In the two-factor, $n = 1$ version of the model, the probability of the τ-period forward rate moving from state (r, s) at time t to state (r, s) at time $t + 1$ is

$$q_{t,\tau,r,s}(1)q_{t,\tau,r,s}(2).$$

$$[1 - q_{t,\tau,r,s}(1)]q_{t,\tau,r,s}(2)$$

is the probability of moving to state $(r+1, s)$, and so on. The value of the caplet at time t in state (r, s) is therefore

$$\begin{aligned} cap_{\tau,t,r,s} &= \{q_{t,\tau,r,s}(1)q_{t,\tau,r,s}(2)cap_{\tau,t+1,r,s} \\ &+ q_{t,\tau,r,s}(1)[1-q_{t,\tau,r,s}(2)]cap_{\tau,t+1,r,s+1} \\ &+ [1-q_{t,\tau,r,s}(1)]q_{t,\tau,r,s}(2)cap_{\tau,t+1,r+1,s} \\ &+ [1-q_{t,\tau,r,s}(1)][1-q_{t,\tau,r,s}(2)]cap_{\tau,t+1,r+1,s+1}\} \left[\frac{1}{1+f_{t,t,r,s}\delta}\right] \end{aligned}$$

where the discounting is at the time t spot rate, f_t, t.

In Table 1, we report caplet prices for a vanilla case where $\delta = 0.25$, the forward rate curve at $t = 0$ is flat 5%, and the caplet volatility structure is flat 20%. Caplet prices are shown for strike rates $k = 5\%$, 6%. The model prices come from computing the LMM over 20 periods (5 years). We first price the caplets using the Black model with implied volatilities of 20% for all the options, using equation (1). These are shown in columns 1 and 4 of the Table. In columns 3 and 6 of the Table we show the model prices for the one-factor case (where $\alpha_{1,1} = 1$. These show significant mis-pricing for short maturities Then in columns 2 and 5 we report similar prices for the two-factor case. In the case of the two-factor model, the allocation constants are $a_{1,0} = 0.8$ and $b = 0.7$. It is clear from the table that in most cases (excluding the very short term caplets in the one-factor case) the pricing of the caplets is consistent with the Black model. This confirms that the drift factors and probabilities are correct.

We now look at the pricing errors as the binomial density, n is increased. In Table 3, we again compare the LMM caplet prices with the Black model prices. This time we choose parameter values: $\delta = 0.5$, the forward rate curve at $t = 0$ is flat 6%, and the caplet volatility structure is flat 20%. Caplet prices are shown for strike rates 6%. Column 2 shows the Black prices. Columns 3 and 4 show

5.2 Swaption Pricing

One of the most important interest-rate options to price is the option to enter an interest-rate swap, or swaption. The payoff on a European-style swaption, with strike rate k, is the value of entering a swap with final maturity date, sm, on the option maturity date τ. The cash flows on a pay-fixed swaption are shown below.

$$
\begin{array}{ccccccccccc}
 & & \text{option maturity} & & \text{reset date} & & \text{reset date} & & & & \text{swap maturity} \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \text{———} & \tau & \text{———} & \tau+\delta & \text{———} & \tau+2\delta & & \cdots & \text{———} & sm \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & & & (f_{\tau,\tau}-k)\delta & & (f_{\tau+\delta,\tau+\delta}-k)\delta & & & & (f_{sm-\delta,sm-\delta}-k)\delta
\end{array}
$$

Let

$$\nu = \frac{sm-\tau}{\delta}$$

be the number of FRAs in the swap. Index these $i = 0, 1, ..., \nu - 1$. Then the ith FRA pays $(f_{\tau+i\delta} - k)\delta$ at time $\tau + i\delta + 1$. We value the right to enter the swap at τ by valuing the individual FRAs in the swap. These in turn are valued by reversing the FRA at times $\tau + i\delta$. The value of the i-th FRA, at time τ, is the given by:

$$fk(\tau, i\delta) = (f_{\tau+i\delta} - k)\delta \frac{1}{1+f_{\tau,\tau}\delta} \frac{1}{1+f_{\tau,\tau+\delta}\delta} .... \frac{1}{1+f_{\tau,\tau+i\delta}\delta}.$$

Here, the reversed payoff on the FRA is discounted back to time τ using the forward rates at time τ. The value of the swap at τ is the sum of the value of the FRAs. This is

$$sk(\tau, sm) = \sum fk(\tau, i\delta).$$

Finally, the payoff on the swaption at τ is

$$swn(\tau, sm) = max[\sum fk(\tau, i\delta), 0].$$

In Table 2 we show results from using a one-factor version of the model to price European-style and 'fixed-tail' Bermudan-style swaptions. The results are directly comparable with those recorded from the LMM, as reported in Anderson (2000). As in Andersen (2000), the Bermudan-style swaptions have a 'lockout period' shown in column four of the table. The prices of the European-style swaptions are very close to those quoted in Andersen (2000), except for the very short maturity options. This is explained by the small number of nodes in the binomial version. In the two cases that can be compared, the HSS-binomial model overprices the Bermudan-style swaptions by a small number (three) of basis points. There are two possible explanations of this excess. First, since the Anderson estimates are lower bounds, it could be that the HSS-binomial prices are the correct ones. However, a more likely explanation is that again the binomial approximation. In this case the HSS-binomial method could be biased upwards. At each node the exercise strategy chooses the highest priced option and if some of these are overpriced due to the binomial density, then these will tend to be chosen. This bias can be reduced by increasing the binomial density.

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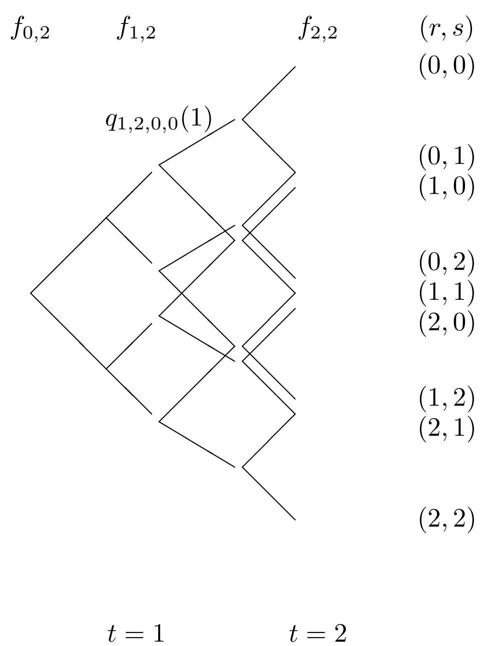
Figure 1
A Recombining Binomial-Tree: Two Factors Case



Notes:

1. The figure illustrates the HSS process for the forward rate maturing at date 2, over the period from time 0 to time 2.

2. The two-factor model produces 4 nodes at $t = 1$, 9 nodes at $t = 2$, and in general $(t + 1)^2$ at time t. The lattice recombines in two dimensions.

3. The volatility of the first factor is $\sigma_1(1) = 5$, $\sigma_2(1) = 4$.

4. The volatility of the second factor is $\sigma_1(2) = 3$, $\sigma_2(2) = 3$.

Table 1: Caplet Prices: Black, One-Factor and Two-Factor Models

	5% Strike			6% Strike		
maturity	(1) Black	(2) 2-factor	(3) 1-factor	(4) Black	(5) 2-factor	(6) 1-factor
1	0.00049	0.00049	0.00061	0.00002	0.00000	0.00000
2	0.00069	0.00067	0.00063	0.00009	0.00009	0.00003
3	0.00083	0.00082	0.00089	0.00017	0.00017	0.00019
4	0.00095	0.00093	0.00092	0.00025	0.00025	0.00019
5	0.00104	0.00103	0.00108	0.00033	0.00032	0.00036
6	0.00113	0.00111	0.00112	0.00041	0.00039	0.00037
7	0.00120	0.00119	0.00122	0.00048	0.00046	0.00050
8	0.00127	0.00125	0.00125	0.00055	0.00052	0.00052
9	0.00133	0.00131	0.00134	0.00061	0.00058	0.00062

1. Caplets assume $\delta = 0.25$.
2. Assumes volatility is 20%, the forward curve is flat 5%.

Table 2: Swaption Prices (basis points): One-Factor Model

European			Bermudan		
option maturity	swap end	price	lockout	swap	price
1	4	115 (122)	1	4	161 (158)
2	4	111 (111)	2	4	125
3	4	67 (66)	3	4	70
2	5	162 (162)	2	5	191 (188)
3	5	130 (128)	3	5	139
4	5	73 (72)	4	5	74

1. All options are on $\delta = 0.5$ reset swaps. Strike price is 6%, volatility is 20%, the forward curve is flat 6%.
2. The prices can be compared with those shown in Andersen (2000), Table 1. These prices are shown in brackets.

Table 3

A Comparison of the HSS-LMM Caplet Prices with the Black Model

	Black	n=2	n = 1	Black-(n=1)	Black-(n=2)	Richardson	Black - Richardson
1	0.001594	0.001479	0.001986	-0.00039	0.00012	0.00097	0.00062
2	0.002187	0.002146	0.002063	0.00012	0.00000	0.00223	-0.00004
3	0.002598	0.002595	0.002789	-0.00019	0.00000	0.00240	0.00020
4	0.00291	0.002929	0.002897	0.00001	-0.00002	0.00296	-0.00005
5	0.003156	0.003187	0.003263	-0.00011	-0.00003	0.00311	0.00004
6	0.003354	0.003391	0.003388	-0.00003	-0.00004	0.00339	-0.00004
7	0.003515	0.003553	0.003562	-0.00005	-0.00004	0.00354	-0.00003
8	0.003645	0.003681	0.003697	-0.00005	-0.00004	0.00366	-0.00002
9	0.00375	0.00378	0.003748	0.00000	-0.00003	0.00381	-0.00006
10	0.003835	0.003857	0.003887	-0.00005	-0.00002	0.00383	0.00001
11	0.003901	0.003913	0.003852	0.00005	-0.00001	0.00397	-0.00007
12	0.003953	0.003951	0.003992	-0.00004	0.00000	0.00391	0.00004

Table 4: Recombining Tree of Forward Rates: Transition Probabilities

f_0_0_0_0	0.050		
f_0_1_0_0	0.050	0.501	0.500
f_0_2_0_0	0.050	0.502	0.500
f_0_3_0_0	0.050	0.503	0.500

f_1_1_0_0	0.057		
f_1_2_0_0	0.057	0.461	0.564
f_1_3_0_0	0.057	0.435	0.567
f_1_1_0_1	0.051		
f_1_2_0_1	0.050	0.461	0.428
f_1_3_0_1	0.049	0.434	0.423
f_1_1_1_0	0.049		
f_1_2_1_0	0.050	0.547	0.564
f_1_3_1_0	0.051	0.579	0.567
f_1_1_1_1	0.043		
f_1_2_1_1	0.043	0.547	0.428
f_1_3_1_1	0.043	0.579	0.423

f_2_2_0_0	0.065		
f_2_3_0_0	0.066	0.422	0.627
f_2_2_0_1	0.058		
f_2_3_0_1	0.057	0.422	0.491
f_2_2_0_2	0.052		
f_2_3_0_2	0.050	0.422	0.355
f_2_2_1_0	0.056		
f_2_3_1_0	0.057	0.507	0.627
f_2_2_1_1	0.050		
f_2_3_1_1	0.050	0.507	0.491
f_2_2_1_2	0.044		
f_2_3_1_2	0.043	0.507	0.355
f_2_2_2_0	0.048		
f_2_3_2_0	0.049	0.593	0.627
f_2_2_2_1	0.042		
f_2_3_2_1	0.043	0.593	0.491
f_2_2_2_2	0.037		
f_2_3_2_2	0.037	0.592	0.355

f_3_3_0_0	0.075
f_3_3_0_1	0.066
f_3_3_0_2	0.059
f_3_3_0_3	0.052
f_3_3_1_0	0.064
f_3_3_1_1	0.057
f_3_3_1_2	0.050
f_3_3_1_3	0.045
f_3_3_2_0	0.054
f_3_3_2_1	0.048
f_3_3_2_2	0.043
f_3_3_2_3	0.038
f_3_3_3_0	0.046
f_3_3_3_1	0.041
f_3_3_3_2	0.036
f_3_3_3_3	0.032