

Pseudo Risk-Neutral Valuation Relationships and the Pricing of Options

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Abstract

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Given an estimate of the rate of increase or decline in the relative risk aversion of the pricing kernel, we price options by deriving an extension of the Black-Scholes-Merton-Rubinstein option pricing model. We build on the analysis of Heston (1993), who derived a missing-parameter option pricing model, generalising his results to two or more parameters. This leads us to derive conditions for a 'generalized' risk-neutral valuation relationship, a relationship between the price of an option and any two contingent claims. This allows us to extend option pricing to cases where the pricing kernel has declining elasticity. In this setting, there exists a risk neutral valuation relationship on a transformed asset. This relationship is termed a pseudo risk-neutral valuation relationship.

1 Introduction

A necessary and sufficient condition for the Black-Scholes model to hold for the pricing of options on a lognormal asset, is that the pricing kernel has constant elasticity. In a representative agent economy, in which the options are written on aggregate wealth, the corresponding necessary and sufficient condition is that the agent has constant relative risk averse (CRRA) utility.¹ The Black-Scholes model is the prime example of a risk-neutral valuation relationship (RNVR). The relationship between the option price and the price of the underlying asset is the same as it would be in a risk-neutral world. However, while the RNVR is at the core of option pricing theory, there is considerable empirical evidence to suggest that the Black-Scholes RNVR does not hold in practice for options on a wide range of assets. As a result, much research effort has been expended on extending the range of RNVRs, using alternative distributional assumptions. However, in spite of these extensions, the resulting models are unlikely to reflect market reality, since there is little reason to expect *any* RNVR to hold in practice.

For example, considering the case of options on aggregate wealth, there is considerable evidence that the economy does not act like a representative agent with CRRA utility. Recent evidence from Jackwerth (2000) and Ogaki and Zhiang (2001), for example, suggests declining relative risk aversion (DRRA). Also, the theoretical models of Benninga and Meyshar (2000) suggest that even if individual investors have CRRA utility, then heterogeneity across investors may induce the representative investor to have DRRA. If the representative investor has DRRA then a RNVR will not hold for the pricing of options on a lognormally distributed wealth.

In this paper we suggest a method for pricing options when RNVRs do not exist. Given an estimate of the rate of increase or decline in the elasticity of the pricing kernel, we price options by deriving an extension of the Black-Scholes-Merton-Rubinstein option pricing model. The estimate of the rate of decline in elasticity may come from empirical evidence. Alternatively, the estimate may come from observing one contingent claim price, for example an at-the-money option price, in addition to the price of the underlying asset.

Recent work by Mathur and Ritchken (2000) and Franke, Stapleton and Subrahmanyam (1999) has discussed the pricing of options under alternative assumptions regarding the risk aversion of the representative investor. Mathur and Ritchken claim that the Black-Scholes price, obtained under CRRA, is the minimum price of an option. Second Franke, Stapleton and Subrahmanyam show that, under DRRA, all options have higher prices than in the

¹In a continuous time economy a sufficient condition for the pricing kernel to have constant elasticity is that the forward price of the market portfolio follows a geometric Brownian motion.

CRRA economy. In this paper we extend these results by showing the *exact* option prices that result from the assumption of DRRA, using assumed knowledge of the rate of decline of risk aversion.

The literature on the pricing of contingent claims in discrete time models goes back to Rubinstein (1976) and Brennan (1979). Rubinstein established the Black-Scholes RNVR for options on a lognormal asset in an economy where the representative investor had CRRA preferences. Brennan found a RNVR in the case of normally distributed assets and CARA preferences. Stapleton and Subrahmanyam (1984) show that similar relationships hold for options on multiple variables. Recent contributions by Camara (2002) have extended the set of known RNVRs further to probability density functions that are functions of the normal density. In spite of this work, there is still no derivation in the literature of the *complete set* of RNVRs. In order to price options on lognormal assets under DRRA we need to select from a wider set of RNVRs. The complete set of RNVRs is a subset of Heston's 'missing parameter' valuation relationships. Heston (1993) derives necessary and sufficient conditions for contingent claims pricing formulae to be independent of a preference parameter. Since, in a RNVR the price of the claim does not depend on the risk aversion parameter, any RNVR must be a member of the larger set of Heston's preference parameter free valuation relationships (PPFVRs).

We begin the analysis, therefore, by exploring within Heston's PPFVRs for the complete set of RNVRs. We then turn our attention to the question of option valuation for cases where RNVRs do not exist. First, we extend Heston's PPFVR to the two-dimensional case, where the relationship between a contingent claim price and any two other claim prices is independent of two preference parameters. This leads to a two-dimensional RNVR, where the relationship between a contingent claim price and any two other claim prices is the same under risk aversion and risk neutrality. We then show that this case is equivalent to the existence of a pseudo risk neutral valuation relationship. By using additional information on the rate of decline in the risk aversion of the pricing kernel, we transform the probability distribution function of the underlying asset and select a RNVR that can price claims on this 'pseudo asset'. The resulting pricing relationship is termed a pseudo risk neutral valuation relationship (PRNVR).

In section 2, we define a RNVR and Heston's 'missing parameter' generalization. In section 3, we derive the necessary and sufficient conditions for the existence of a RNVR. These results extend the set of known RNVRs in the literature. In particular, we find conditions for the existence of a RNVR on a generalized lognormal asset. In section 4, we extend Heston's concept of a preference-parameter-free valuation relationship (PPFVR) to two dimensions and use it to price options with a two-dimensional RNVR. Then, we define the set of pseudo risk-neutral valuation relationships. The key idea here, is that although no RNVR exists

for the asset on which the option is written, there exists a RNVR for a transformed asset in an alternative economy. In section 5, we apply this idea of a PRNVR to the pricing of options on log-normal assets, for the case when the pricing kernel has declining elasticity. Given an estimate of the rate of decline in elasticity we price European-style options using a PPFVR. Section 6 concludes the paper.

2 RNVRs and Heston's Generalization

We consider the valuation of European-style contingent claims, with maturity T , paying $c_T(x)$, which depend on an underlying asset with payoff x_T . For convenience we write $x_T \equiv x$. We assume a complete market economy in which the forward price of a European-style contingent claim on x , paying $c_T(x)$ is given by

$$c_{0,T} = E_0[c_T(x)\phi(x)], \quad (1)$$

where $\phi(x)$ is the asset specific pricing kernel. Also, the forward price of the underlying asset is given by

$$x_0 = E_0[x\phi(x)]. \quad (2)$$

Following Heston (1993), we assume that the pricing kernel takes the form

$$\phi = \phi(x; \gamma) \quad (3)$$

where γ is a preference parameter.²

First, we define the concepts of risk-neutral valuation of contingent claims and Heston's generalization to 'missing parameters' valuation.

Definition 1 [*A Risk-Neutral Valuation Relationship*]

A risk-neutral valuation relationship (RNVR) exists for the valuation of contingent claims on an asset if the relationship between the price of the claim in (1) and the price of the asset in (2) is the same as it would be under risk neutrality.

If a RNVR exists, it follows that the formula for the price of the contingent claim can be written as a function of x_0 , where the preference parameter γ is irrelevant. In the literature,

²We abstract from consideration of the discounting of the contingent claim by working with the forward price of the claim. Hence ϕ is a 'forward pricing kernel'. Thus we write $\phi = \phi(x; \gamma)$, instead of Heston's $\phi = \beta\phi(x; \gamma)$, where β is a discount factor.

well known RNVRs are the Black-Scholes formula for options on assets with log-normal distributions and the Brennan (1979) formula for the price of an option on a normally distributed asset. Other RNVRs applying to assets following a displaced diffusion process and to options on multiple assets have been established by Rubinstein (1983), Stapleton and Subrahmanyam (1984a), and Camara (2002). However, the set of known RNVRs is limited both by the type of probability distribution function assumed and by the form of the pricing kernel required to derive a RNVR, given the probability distribution. This has led Heston (1993) to propose a generalization (of the set of RNVRs). Heston proposes a set of contingent claims formulae which are independent of the preference parameter γ and of one parameter of the probability distribution. To be precise he assumes that the probability distribution function of x ‘belongs to a family of densities that depend on an additional parameter’ and defines what we may term a

Definition 2 [*Preference-Parameter-Free Valuation Relationship*]

A Preference-Parameter-Free Valuation Relationship (PPFVR) exists for contingent claims on an asset, if the relationship between the forward price of the claim and the forward price of the asset is independent of the preference parameter, γ .

Heston establishes necessary and sufficient conditions for the existence of PPFVRs. We have:

Lemma 1 [*Necessary and Sufficient Conditions for a Preference-Parameter-Free Valuation Relationship*]

Given the pricing kernel in equation (3) and a probability density function $f(x, q)$, the pricing of contingent claims is independent of q and γ if and only if the pricing kernel and the probability density function have the form

$$\phi(x; \gamma) = b(\gamma)h(x)e^{\gamma k(x)}$$

and

$$f(x; q) = a(q)g(x)e^{qk(x)}. \tag{4}$$

Proof: Heston proves a similar result for the spot prices of contingent claims, see Heston p 946-7. The same method of proof for the case of forward prices of claims yields the Lemma. \square

Note that the set of PPFVRs includes the set of RNVRs. This must be the case, since the parameter representing risk aversion drops out in the case of a RNVR. In the following section we search for the set of possible RNVRs. Since these must be within the wider set of Heston's PPFVRs, we restrict ourselves to assets with probability density functions of the form in equation (4).

3 Option Pricing in Discrete Time Models: Conditions for the Existence of Risk-Neutral Valuation Relationships.

Option pricing models typically assume that the type of distribution of x is known (for example lognormal) and that the type of function $\phi(x)$ is also known (for example, a declining power function)³. When strong enough assumptions are made it is possible to use the price of the underlying asset to back out the risk-neutral density and establish a RNVR.

In this section we assume a general class of distributions of the form required by Lemma 1

$$f(x; q) = ag(x)e^{qk(x)}. \quad (5)$$

where a is a constant, q is a parameter of the distribution, $g(x)$ is a positive function, and where $k(x)$ is monotonic. We take this class, since from Heston (1993), RNVRs cannot exist for contingent claims on assets with probability distributions outside this set. As noted by Heston, this set leaves considerable flexibility in the specification of the probability density. The following examples illustrate this. Examples of this class are:

1. x is Normal:

$$f(x; q) = n(x; \mu) = ag(x)e^{\hat{\mu}k(x)} \quad (6)$$

where

$$\begin{aligned} a &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\mu^2/2\sigma^2} \\ g(x) &= e^{-x^2/2\sigma^2} \\ k(x) &= x \\ \hat{\mu} &= \frac{\mu}{\sigma^2} \end{aligned}$$

³The typical assumption made in continuous time models is that the asset price follows a geometric Brownian motion. This implies that the pricing kernel is a declining power function

This is one of the distributions assumed by Brennan (1979) in his establishment of a RNVR using CARA utility.

2. x is Log-normal:

$$f(x; q) = n(\ln x; \mu)/x = ag(x)e^{\hat{\mu}k(x)} \quad (7)$$

where

$$\begin{aligned} a(\mu) &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\mu^2/2\sigma^2} \\ g(x) &= \frac{e^{-(\ln x)^2/2\sigma^2}}{x} \\ k(x) &= \ln x \\ \hat{\mu} &= \frac{\mu}{\sigma^2} \end{aligned}$$

This is the distribution assumed by Rubinstein (1976) in his establishment of a RNVR using CRRA utility.

3. x is 'Fat-tailed'⁴

$$\begin{aligned} f(x; \mu, \sigma, \alpha, \beta) &= ax^{-\alpha}n(\ln x; \mu, \sigma), \quad x < 1 \\ &= ax^{\beta}n(\ln x; \mu, \sigma), \quad x > 1 \end{aligned} \quad (8)$$

where

$$a = 1/[e^{((-\frac{\mu}{\sigma^2}\alpha + \frac{1}{2}\alpha^2)\sigma^2)}N(-(-\alpha + \frac{\mu}{\sigma^2})\sigma) + e^{((\frac{\mu}{\sigma^2}\beta + \frac{1}{2}\beta^2)\sigma^2)}N((\beta + \frac{\mu}{\sigma^2})\sigma)],$$

which is a normalization factor.

4. x is 'generalized lognormal':

$$f(x; q) = a(q)\hat{g}(x)e^{qk(x)}. \quad (9)$$

with

$$\begin{aligned} a &= \frac{1}{\int \hat{g}e^{qk(x)}} \\ \hat{g}(x) &= \frac{e^{-(\ln x)^2/2\sigma^2}}{x}G(x) \\ k(x) &= \ln x \end{aligned}$$

where $G(x)$ is any positive function of x .

⁴An explanation of why this distribution is fat-tailed is given in the appendix.

As can be appreciated from these examples, the set of distributions is quite broad. Further, the distributions are not necessarily continuous in x . We now consider the necessary and sufficient conditions for the existence of RNVRs for assets with distributions in this set.

First, we formalise this idea of a RNVR. Above, we defined a RNVR in terms of the forward prices of the asset and of the contingent claim. More formally, in the case of the set of distributions introduced above, we can characterize the RNVR as follows. Since, under risk neutrality the time 0 forward price of the asset would be given by the expected value:

$$x_0 = \int x f(x; q) dx, \quad (10)$$

and since $f(x)$ is of the form

$$f(x; q) = a(q)g(x)e^{qk(x)}. \quad (11)$$

we can solve (10) for the parameter q . If we call this ‘risk neutral’ value of q , q_0 , then a RNVR exists for the value of a contingent claim if and only if

$$c_{0,T} = \int c_T(x) f(x; q_0) dx. \quad (12)$$

This characterization provides us with an operational definition of a RNVR. We can now derive conditions for the existence of a RNVR. The following proposition provides the complete set of possible RNVRs.

Proposition 1 [*Necessary and Sufficient Conditions for a Risk-Neutral Valuation Relationship*]

Assume that the distribution of x , the payoff of the underlying asset, is of the form

$$f(x; q) = ag(x)e^{qk(x)}.$$

where q is a parameter and $k(x)$ is monotonic function. Assume there exist complete and arbitrage free markets for contingent claims on x . Then there exists a RNVR for contingent claims on x , if and only if the pricing kernel $\phi(x)$ is of the form

$$\phi(x; \gamma) = \alpha e^{\gamma k(x)}.$$

Proof

Sufficiency

Since

$$\phi(x) = \alpha e^{\gamma k(x)}$$

and $E[\phi(x)] = 1$, we have

$$\int_0^{+\infty} \alpha a(q) g(x) e^{(q+\gamma)k(x)} dx = 1,$$

and it follows that

$$\alpha a(q) = a(q + \gamma).$$

Hence we can write

$$f(x; q)\phi(x) = a(q + \gamma)g(x)e^{(q+\gamma)k(x)} = f(x; q + \gamma).$$

Since the forward price of the underlying asset is given by

$$\int_0^{+\infty} x f(x; q)\phi(x) dx = x_0,$$

we obtain

$$\int_0^{+\infty} x f(x; q + \gamma) dx = x_0.$$

From this we can infer $f(x; q + \gamma)$ and, hence, $q + \gamma$. The forward price of a contingent claim paying $c_T(x)$ is

$$c_{0,T} = \int_0^{+\infty} c_T(x)\phi(x)f(x; q)dx = \int_0^{+\infty} c_T(x)f(x; q + \gamma)dx. \quad (13)$$

The contingent claim price in (13) is given by (12) with $q_0 = q + \gamma$ and a RNVR exists.

Necessity

Since a RNVR exists for every contingent claim, it follows that

$$\phi(x) = f(x; q_0)/f(x; q).$$

Since any RNVR is also a PPFVR, it follows from Heston (1993, Proposition 1) that we can write

$$f(x; q) = a(q)g(x)e^{qk(x)}.$$

Hence,

$$\begin{aligned}\phi(x) &= \frac{\alpha(q_0)}{\alpha(q)} e^{(q_0-q)k(x)} \\ &= \alpha e^{\gamma k(x)}\end{aligned}$$

□

We now apply the proposition to our previous examples of PDFs:

1. If x is Normal as in (6) then a RNVR exists iff

$$\phi(x; \gamma) = \alpha e^{\gamma x}.$$

If x is the aggregate wealth, Brennan (1979) shows that this pricing kernel is equivalent to a representative agent economy where the agent has constant absolute risk aversion.

2. If x is Log-normal as in (7), then a RNVR exists iff

$$\phi(x; \gamma) = \alpha e^{\gamma \ln x} = \alpha x^\gamma.$$

If x is the aggregate wealth, Rubinstein (1976) shows that this pricing kernel is equivalent to a representative agent economy where the agent has constant relative risk aversion.

3. x is 'Fat-tailed' as in (8), then a RNVR exists if and only if

$$\phi(x; \gamma) = \alpha x^\gamma.$$

Note that the same conditions: constant elasticity of the pricing kernel guarantees a RNVR for this fat-tailed distribution.

4. x is 'generalized lognormal' as in (9), then a RNVR exists if and only if

$$\phi(x; \gamma) = \alpha x^\gamma.$$

Note that the same conditions: constant elasticity of the pricing kernel guarantees a RNVR for this generalized log-normal distribution.

An Interpretation of Proposition 1

Proposition 1 characterizes the set of possible RNVRs that can exist for the pricing of European-style contingent claims. We now investigate how large this set is. The function $k(x)$ is common to both the probability distribution of the asset and to the pricing kernel. It is this commonality that restricts the range of possible RNVRs. It is easy to establish the following corollaries of Proposition 1.

Corollary 1 [Characterization of $k(x)$]

Assume that

$$f(x; q) = a(q)g(x)e^{qk(x)}.$$

Let $R(x) = \frac{-\phi'(x)}{\phi(x)}$ be defined as the absolute risk aversion of the pricing kernel. Then a RNVR exists for the pricing of contingent claims if and only if

$$k(x) = - \int \frac{R(x)}{\gamma} dx$$

For example, if $R(x)$ is a constant (CARA), then $k(x)$ is linear in x . Corollary 1 allows us to specify exactly when a RNVR will exist. For example, if $R(x) = a/x$ (CRRA), $k(x) = \ln x$ and the Black-Scholes RNVR applies. However, if $R(x) = \frac{1}{x+a}$, $k(x) = \ln(x+a)$ a RNVR may exist if x is a shifted lognormal with a threshold parameter a .

Corollary 2 [Absolute Risk Aversion with Respect to $k(x)$]

Assume that

$$f(x; q) = a(q)g(x)e^{qk(x)}.$$

Define the absolute risk aversion of the pricing kernel with respect to $k(x)$ as

$$\eta(x) = - \frac{\frac{\partial \phi}{\partial k(x)}}{\phi}$$

Then a RNVR exists if and only if

$$\eta(x) = -\gamma,$$

a constant.

The limited set of RNVRs is shown also by Corollary 2. Suppose for example that we wish to price an option on a lognormal asset, where $k(x) = \ln x$. However, suppose that we suspect that the pricing kernel has declining elasticity with respect to $\ln x$. Then it follows from Corollary 2 that no RNVR exists. In order to price options in such economies, we need to generalize the concept of the RNVR. This is the task of the following section.

Finally, we state a relationship between the RNVR and the more general PPFVR of Heston. If $h(x) = 1$ in the pricing kernel:

$$\phi(x; \gamma) = b(\gamma)h(x)e^{\gamma k(x)}$$

then

$$\phi(x; \gamma) = b(\gamma)e^{\gamma k(x)}$$

and according to Proposition 1 a RNVR exists. Hence we have

Corollary 3 [*RNVR and PPFVR*]

Assume that a PPFVR exists and that $h(x) = 1$, then a RNVR exists for the valuation of contingent claims.

The corollary emphasises the obvious point that if there is only one unknown preference parameter and one parameter is invisible, then a RNVR must exist. The problem addressed in the following section is that in reality it is unlikely that there will be only one unknown parameter in the pricing kernel.

4 A Generalized Set of Risk-Neutral Valuation Relationships

The set of RNVRs is severely restricted, as shown by the corollaries above. In order to price contingent claims in less restricted economies, where the pricing kernel is more complex, we need to relax the conditions imposed on the pricing kernel. For example, how can we price an option on a log-normally distributed asset, if the pricing kernel has non-constant elasticity? The answer may lie in a generalization of Heston's concept of a preference-parameter-free valuation relationship (PPFVR).

Note that the set of PPFVRs includes the set of RNVRs. This must be the case, since the parameter representing the mean drops out in the case of a RNVR. However, the set of PPFVRs is generally too broad for the purposes of option valuation. Just because one

preference parameter drops out of the option pricing formula, does not imply that valuation is possible. Options can only be valued if *all* the other parameters affecting the pricing kernel are known. This motivates the definition of a subset of PPFVRs, in which these other parameters are known or disappear.

First, we consider the subset of PPFVRs where *two* preference parameters are invisible in the option price. We define:

Definition 3 [*Two-Dimensional Preference-Parameter-Free Valuation Relationship*]

A two-dimensional preference-parameter-free valuation relationship (TPPFVR) exists for the price of contingent claims if the relationship between the forward price of the claim and the forward price of two other contingent claims is independent of two preference parameters, γ_1 and γ_2 .

The set of TPPFVRs defines a subset of PPFVRs, since if one parameter is invisible this clearly does not imply that two parameters are invisible. A straightforward generalization of Heston's Proposition 1 (our Lemma 1) then yields:

Lemma 2 [*Necessary and Sufficient Conditions for a Two-Dimensional Preference-Parameter-Free Valuation Relationship*]

Given the pricing kernel $\phi(x) = \phi(x : \gamma_1, \gamma_2)$ and a probability density function $f(x)$, the relationship between the price of a contingent claim and two other claims is independent of γ_1 and γ_2 if and only if the pricing kernel and the probability density function have the form

$$\phi(x; \gamma) = b(\gamma)h(x)e^{\gamma_1 k_1(x) + \gamma_2 k_2(x)}$$

and

$$f(x) = a(q)g(x)e^{q_1 k_1(x) + q_2 k_2(x)}. \quad (14)$$

Proof.

We take the case where one of the claims is the asset itself, with forward price, x_0 and the other claim is a claim with price $c_{0,a}$. If the pricing kernel and the PDF have the above form, then

$$\phi(x)f(x) = abh(x)g(x)e^{(\gamma_1 + q_1)k_1(x) + (\gamma_2 + q_2)k_2(x)}.$$

Given the forward prices:

$$x_0 = \int_0^{+\infty} x f(x) \phi(x) dx,$$

$$c_{0,a} = \int_0^{+\infty} c_a(x) \phi(x) f(x) dx$$

we can solve for $(\gamma_1 + q_1) = m(x_0, c_{0,a})$ and $(\gamma_2 + q_2) = n(x_0, c_{0,a})$. Any other claim then has a price

$$c_{0,b} = \int_0^{+\infty} c_b(x) \phi(x) f(x) dx = abh(x)g(x)e^{m(x_0, c_{0,a})k_1(x) + n(x_0, c_{0,a})k_2(x)},$$

which is independent of γ_1 and γ_2 . Necessity is shown in the appendix. \square

The importance of the TPPFVR is that it implies a two-dimensional risk-neutral valuation relationship when $h(x) = 1$, i.e. when only two parameters determine the pricing kernel. First we define:

Definition 4 [*A Two-Dimensional-Risk-Neutral Valuation Relationship*]

A two-dimensional risk-neutral valuation relationship (TRNVR) exists for the pricing of contingent claims on an asset, if the price of a claim can be written as a function of the prices of the underlying asset and another option on the asset, in a manner that is the same under risk aversion and risk neutrality.

As in the one-dimensional case, we interpret prices in the definition as forward prices.

Proposition 2 [*Necessary and Sufficient Conditions for a Two-Dimensional-Risk-Neutral Valuation Relationship*]

Assume that Lemma 2 holds, then a two-dimensional risk-neutral valuation relationship (TRNVR) exists for the valuation of contingent claims on x if and only if $h(x) = 1$.

Some care is required in interpreting the TRNVR. Although the relationship is the same as under risk neutrality it is not a RNVR in the usual sense. It is a valuation relationship that requires both the underlying asset and the *value of an additional contingent claim* in order to price any other claim. The second claim required could be an at-the-money call option, for example. If a TRNVR exists, any option can be priced if we know the price of the underlying asset and the price of an at-the-money call option. We now present an alternative interpretation of the pricing. First, we define what we term a *pseudo* risk-neutral valuation relationship:

Definition 5 [*Pseudo Risk-Neutral Valuation Relationship*]

A pseudo risk-neutral valuation relationship (PRNVR) exists for contingent claims on an asset x with distribution $f(x)$, if a RNVR exists for contingent claims on a 'pseudo' asset with transformed distribution $f_1(x)$, given a transformed pricing kernel $\phi_1(x)$.

It follows immediately from the conditions for a RNVR, that a PRNVR exists if the transformed pricing kernel has certain properties. In fact, we find:

Proposition 3 [*Necessary and Sufficient Conditions for a Pseudo-Risk-Neutral Valuation Relationship*]

Assume that

$$f(x; q) = a(q)g(x)e^{qk(x)}.$$

and that the pricing kernel is a separable function of the form

$$\phi(x; \gamma) = \hat{\phi}(x; \gamma)h(x).$$

where $h(x)$ is a known function.

A pseudo-risk-neutral valuation relationship exists for the valuation of contingent claims on x if and only if $k(x)$ is monotonic and the pricing kernel is of the form:

$$\phi(x) = b(\gamma)h(x)e^{\gamma k(x)},$$

Proof

The transformed asset has a generalized lognormal distribution as in example 4 above. From Proposition 1, a RNVR exists for the transformed asset and the transformed pricing kernel if and only if

$$\phi_1(x) = b_1(\gamma)e^{\gamma k(x)}.$$

This implies that the pricing kernel has the form in the Proposition. \square

Assuming that the pricing kernel is a separable function, the Proposition states that a PRNVR exists for the pricing of claims if and only if $k(x)$ is monotonic and the Heston conditions hold. We can then price the claim as follows. Given the form of the distribution

function and the pricing kernel for the transformed asset, a RNVR exists and the option valuation is independent of γ and q . The option price is given by

$$\int c_T(x) a(q) g(x) e^{(q+\gamma)k(x)} b(\gamma) h(x) dx \quad (15)$$

where

$$q + \gamma$$

is found from the solution of

$$x_{0,T} = \int x a(q) g(x) e^{(q+\gamma)k(x)} h(x) dx.$$

Note that the conditions for the PRNVR in Proposition 3 are the same as those for the TRNVR in Proposition 2. In order to price options using Proposition 2, we need two contingent claim prices, for example, the forward price of the asset and the forward price of an at-the-money call option on the asset. In order to price options using Proposition 3, we need one contingent claim price, for example, the forward price of the asset and knowledge of the function $h(x)$. This knowledge could come from observation of another contingent claim price. However, alternatively if no other prices are observable, knowledge of $h(x)$ could come from empirical estimation of the properties of the pricing kernel. In the case of options on the aggregate wealth, this might come, for instance, from empirical estimation of the rate of decline in risk aversion of the representative investor.

5 An Application of the Pseudo-Risk-Neutral Valuation Relationship to the Pricing of Options on Log-normal Assets

In this section we price European-style options on an asset which has a lognormal distribution, as defined in example 2, in section 3. We also assume that the pricing kernel exhibits positive, declining elasticity and that *the rate of decline is known*. The pricing kernel has the form:

$$\phi(x) = b e^{k_2(x)} e^{\gamma k(x)},$$

where $k(x) = \ln x$. and where b is given by the condition

$$E[\phi(x)] = \int b e^{k_2(x)} a g(x) e^{(\mu+\gamma) \ln x} dx = 1.$$

In this case, if $k_2(x) = 0$, the elasticity of the pricing kernel is

$$-\frac{\partial \ln[\phi(x)]}{\partial \ln x} = -\gamma,$$

and a RNVR exists. However, we now assume that $k_2(x)$ is declining in x . In this case, the function $k_2(x)$ is separable and given the form of the transformed pricing kernel:

$$\phi_1(x) = b_1 e^{\gamma k(x)},$$

a pseudo-risk-neutral valuation relationship exists, by Proposition 3.

The resulting option pricing model is calibrated using estimates of volatility and of the rate of decline of the pricing kernel's elasticity. The resulting option prices exceed those predicted by the Black-Scholes model. In the following, we assume a specific form for $k_2(x)$.

An Example

We assume that $k_2(x) = x^{-1}$. In this case

$$\frac{\partial k_2(x)}{\partial x} = -\frac{1}{x^2}$$

and the elasticity of ϕ is

$$-\frac{\partial \ln \phi}{\partial \ln x} = \frac{1}{x} - \gamma.$$

Hence, in this case the elasticity is declining in x .

In order to incorporate both the cases of declining and constant relative risk aversion, we assume the pricing kernel has the form:

$$\phi = b e^{y/x} e^{\gamma \ln x}, \quad y = 0 \text{ or } 1.$$

Hence with $y = 0$, ϕ is CRRA and with $y = 1$, ϕ is DRRA. To ensure that ϕ has the required property $E(\phi) = 1$, we first solve for $b(\gamma)$:

$$b(\gamma(y)) = \frac{1}{E[e^{y/x} e^{\gamma \ln x}]}, \quad y = 0, 1.$$

By no arbitrage, the forward price of the underlying asset is given by

$$x_0 = E(x\phi).$$

In order to make meaningful comparisons, using different pricing kernels, we hold the asset forward price constant, as in Franke, Stapleton and Subrahmanyam (1999). First, we assume CRRA ($y = 0$) and solve

$$x_0 = E[xb(\gamma(0))e^{0/x}e^{\gamma(0)\ln x}]$$

for $\gamma(0)$. Then we solve

$$x_0 = E[xb(\gamma(1))e^{1/x}e^{\gamma(1)\ln x}]$$

for $\gamma(1)$. Finally, we find the prices of call options under the two pricing kernels, using

$$c_{0,T} = E[c_T(x)\phi].$$

A Numerical Example

In this example, we approximate a lognormal distribution of the underlying asset price with a log-binomial distribution. Initially, in order to benchmark the model, we assume that the expected price of the underlying asset is known. We proceed to price the options assuming first the CRRA pricing kernel and then the DRRA pricing kernel, calibrated to give the same forward price of the underlying asset. We then show that the same option prices are obtained regardless of the mean of the underlying asset. We assume the following data:

1. Time to maturity, $T = 1$ year.
2. Mean of underlying asset price, $E(x) = 1$
3. Volatility of asset price, $\sigma = 0.20$, or 0.30
4. The strike price varies between 0.6 and 1.6 .
5. The asset forward price is $x_0 = 0.9486$, for $\sigma = 0.20$, and $x_0 = 0.8916$, for $\sigma = 0.30$.

In order to derive the option forward prices, we first approximate the log-normal distribution of x with a log-binomial. We use the method of Ho, Stapleton and Subrahmanyam (1995). The details of the methodology are described in the appendix. Using a binomial density of $n = 10$, the lognormal distribution is approximated with a multiplicative binomial distribution with 11 nodes. These states are shown in column 1 of Table 1. The values of the pricing kernel in the case of CRRA and DRRA of the pricing kernel are shown in columns 3 and 4 of the Table. Since x_0 is assumed to be the same under the two pricing kernels, γ must be varied so as to generate an invariant x .

In Table 2 we show the prices of various European-style call options. The results for a zero strike price show that the pricing kernels are calibrated to produce the same forward price

of x . Note that all the DRRA option prices are greater or equal to those under the CRRA pricing kernel, as expected from the results of Franke, Stapleton and Subrahmanyam (1998).

Having priced the options using the explicit pricing kernels under CRRA and DRRA, we now re-price the options using the RNVR (in the case of CRRA) and the PRNVR (in the case of DRRA). In the case of CRRA, the procedure is straightforward. We first approximate the lognormal distribution of x , assuming an expected value equal to the forward price. The resulting density is shown in Table 3 for the case where $\sigma = 30\%$ and $x_0 = 0.8916$. We then price the options using the binomial probabilities shown in the table. The option prices are shown in Table 4. As expected these prices are the same as those in Table 2, to the third decimal place.

In the case of DRRA, we first approximate using a binomial distribution with $E(x) = 0.8916$. We then transform the distribution using the separable function $e^{1/x}$, adjusting the probabilities to ensure that they sum to unity. We then rescale the distribution so that the forward price is maintained. The resulting distribution is shown in Table 3. Finally, we price the options using the risk-neutral distribution. The resulting option prices are shown in Table 4. Again they are virtually the same as those (for DRRA) in Table 2. This is to be expected given the existence of a PRNVR.

6 Conclusions

The risk-neutral valuation relationship is at the core of option pricing theory. However, as shown by our Proposition 1, the set of RNVRs is extremely limited. For example, if the underlying asset has a log-normal distribution, a necessary condition for a RNVR is that the pricing kernel has constant elasticity. When looking for new tractible option pricing models, a more promising approach is to develop Heston's idea of a preference-parameter-free valuation relationship. However, the set of PPFVRs is too large for practical application. To obtain preference-free valuation we need to restrict this set. In this paper, we have worked with those cases where only one preference parameter is unknown. In these cases option prices can be computed given the functional form of the pricing kernel and given the forward price of the underlying asset. We term these pseudo risk-neutral valuation relationships.

Since the log-normal distribution is the most common distribution assumed for financial assets, we have concentrated on using PRNVRs to extend option pricing to the case of non-constant relative risk aversion of the pricing kernel. We have shown that if relative risk aversion is declining in proportion to the underlying asset payoff, option prices can be derived. The resulting option prices are greater, for both puts and calls, than those obtained under constant relative risk aversion. The effect of declining relative risk aversion is particularly strong in the case of out-of-the-money put options. When volatility is high (30%), the effect is to increase the implied volatility to as much as 33%.

Although we have concentrated on examples extending the Black-Scholes RNVR to account for non-constant elasticity of the pricing kernel, the methodology could be applied to generalize the Brennan (1979) normal distribution option pricing model. Here we would assume non-constant *absolute* risk aversion. Similarly, extensions are possible for the Rubinstein (1983) displaced diffusion model. Also, the gamma distribution model of Heston (1993) admits to a similar extension, as does the 'fat-tailed' distribution introduced in this paper.

In this paper we have restricted the analysis to single period models, since these are sufficient for the valuation of European-style options. For results on Bermudan-style and American-style options, multi-period and in the latter case continuous-time models, are required. In these cases results in Huang (2000) suggest that the set of RNVRs is even more restricted (to Brownian motion and geometric Brownian motion). We have also restricted our attention to continuous-state space probability distributions. While our analysis could be extended to cover binomial distributions, as assumed in Stapleton and Subrahmanyam (1984b), for example, the continuous-state space cases considered here appear to be the more relevant.

The PRNVR is equivalent to the pricing of options using a two-dimensional RNVR. The TRNVR relates a contingent claim price to any two other contingent claim prices on the same underlying asset. This analysis admits to a further extension. We could price claims using an n -dimensional RNVR. However, as the number of claims used in the pricing increases, we are in effect merely describing the shape of the pricing kernel in more detail, as in the empirical studies of Jackwerth (2000). The key to option pricing is to price contingent claims with as few pieces of information as possible. However, the analysis here suggests a technique for extending no-arbitrage pricing of claims from the rather unrealistic search for one-dimensional RNVRs of the Black-Scholes type, to the less demanding and hopefully more available set of two-dimensional RNVRs.

7 Appendix

1. A Fat-tailed Distribution for the Asset Price

Let S be the price of a stock at time T and S_0 be its price at time 0. Write $x = S/S_0$. The Black-Scholes option pricing model assumes that x follows a lognormal distribution with density function

$$n(\ln x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}.$$

However, there is evidence that the distributions of stock prices often have fatter tails than lognormal distributions. To capture fat tails, we multiply the lognormal density by $(xe^{-(\mu+\frac{1}{2}\sigma^2)})^{-\alpha}$ for $x < 1$ and the lognormal density by $(xe^{-(\mu+\frac{1}{2}\sigma^2)})^\beta$ for $x > 1$, where α and β are positive. Apparently the factor used to multiply the lognormal density is decreasing in x when $x < 1$ and increasing in x when $x > 1$. This implies that the change made to the lognormal density increases the probability weight to the left and right tails. The values of α and β capture the fatness of the left and right tails. The fatter the tails are, the larger these two parameters. Thus α and β can be called the left and right fat tail indices respectively. The probability distribution density can be written as

$$\begin{aligned} f(x; \mu, \sigma, \alpha, \beta) &= ax^{-\alpha}n(\ln x; \mu, \sigma), \quad x < 1 \\ &= ax^\beta n(\ln x; \mu, \sigma), \quad x > 1 \end{aligned}$$

where

$$a = 1/[e^{((-\frac{\mu}{\sigma^2}\alpha + \frac{1}{2}\alpha^2)\sigma^2)}N(-(-\alpha + \frac{\mu}{\sigma^2})\sigma) + e^{((\frac{\mu}{\sigma^2}\beta + \frac{1}{2}\beta^2)\sigma^2)}N((\beta + \frac{\mu}{\sigma^2})\sigma)],$$

When the two fat-tail indices approach zero, i.e., $\alpha, \beta \rightarrow 1$, we have $a(\sigma, \alpha, \beta) \rightarrow 0$ and $f(x; \mu, \sigma, \alpha, \beta)$ approaches the lognormal density $LN(x; \mu, \sigma)$.

2. Binomial Approximation

The lognormal distribution is approximated with a multiplicative binomial distribution, with proportionate up-move, u and down-move d . The probability of an up-move is $p = 0.5$. We choose

$$\begin{aligned} d &= \frac{1}{e^{2\sigma\sqrt{\frac{1}{n}}} + 1}, \\ u &= 2 - d. \end{aligned}$$

Then x is approximated by

$$x = E(x)u^{n-r}d^r.$$

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Table 1 : The Pricing Kernel Under CRRA and DPRA

state	x	CRRA, $\gamma\sigma^2 = -0.0528(-0.115)$	DRRA, $\gamma\sigma^2 = -0.01(-0.01)$
0	1.84 (2.47)	0.42 (0.28)	0.50 (0.42)
1	1.63 (2.04)	0.50 (0.35)	0.56 (0.47)
2	1.43 (1.69)	0.59 (0.45)	0.62 (0.53)
3	1.26 (1.40)	0.69 (0.57)	0.71 (0.61)
4	1.11 (1.16)	0.82 (0.73)	0.81 (0.72)
5	0.98 (0.96)	0.97 (0.93)	0.95 (0.89)
6	0.86 (0.79)	1.14 (1.18)	1.12 (1.12)
7	0.76 (0.65)	1.35 (1.51)	1.35 (1.50)
8	0.67 (0.54)	1.59 (1.92)	1.67 (2.10)
9	0.59 (0.45)	1.88 (2.45)	2.10 (3.16)
10	0.52 (0.37)	2.23 (3.12)	2.73 (5.15)

1. The table shows the asset value x for a binomial approximation of a log-normal distribution. The approximation has a $n = 10$ binomial steps. The state is represented by the number of down-moves of the process.
2. The volatility is $\sigma = 0.20$ and $\sigma = 0.30$ (numbers for 30% are in brackets). The expected value of the underlying asset payoff is $E(x) = 1$.
3. The pricing kernels are computed using asset forward prices of 0.9846 (vol 20%) and 0.8916 (vol 30%)

Table 2 : The Price of Call Options Under CRRA and DPRA, Volatility 20%

strike price	Call (CRRA)	Call (DRRA)	Call (CRRA)	Call (DRRA)
	$\sigma = 20\%$	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 30\%$
0.0	0.9486	0.9486	0.8916	0.8917
0.6	0.3490	0.3490	0.3002	0.3024
0.8	0.1683	0.1692	0.1530	0.1580
1.0	0.0546	0.0561	0.0677	0.0730
1.2	0.0134	0.0142	0.0265	0.0302
1.4	0.0021	0.0023	0.0092	0.0112
1.6	0.0002	0.0003	0.0034	0.0043
$b(\gamma)$	0.9406	0.3393	0.8775	0.3094

1. The table shows the forward prices of European-style call options on x at various strike prices, which are shown in the first column.
2. The models using the CRRA and DRRA pricing kernels are calibrated to give the same forward price for the underlying asset. This forward price is shown by the option with zero strike price, in the first row.
3. Prices are computed using a binomial approximation with $n = 10$ and $n = 11$ sub-periods, and taking the average of the two approximations.
4. Prices are computed using the pricing kernels from Table 1 for $n = 10$ and corresponding kernels for $n = 11$.

Table 3 : The 'Risk-Neutral' Distribution Under CRRA and DPRA

state	$\sigma = 0.30$ CRRA	prob CRRA	$\sigma = 0.30$ DRRA	prob DRRA
0	2.20	0.001	2.45	0.000
1	1.82	0.010	2.02	0.005
2	1.51	0.044	1.67	0.025
3	1.25	0.117	1.38	0.075
4	1.03	0.205	1.14	0.153
5	0.85	0.245	0.95	0.220
6	0.71	0.205	0.78	0.229
7	0.58	0.117	0.65	0.171
8	0.48	0.044	0.54	0.088
9	0.40	0.010	0.44	0.029
10	0.33	0.001	0.37	0.005

1. The table shows the asset value x for a binomial approximation of a log-normal risk-neutral distribution, where the expected asset price is equal to the forward price, 0.8916. The approximation has a $n = 10$ binomial steps. The state is represented by the number of down-moves of the process.
2. The volatility is $\sigma = 0.30$.
3. In the case of DRRA the model probabilities are adjusted using the rate of decline of the elasticity of the pricing kernel, to derive the transformed distribution for the asset. The payoffs are re-scaled to yield a forward price of 0.8916.

Table 4 : The Price of Call Options Under CRRA and DPRA Using RNVR and PRNVR

strike price	Call (CRRA)	Call (DRRA)
	$\sigma = 30\%$	$\sigma = 30\%$
0.0	0.8916	0.8916
0.6	0.3012	0.3022
0.8	0.1540	0.1584
1.0	0.0674	0.0729
1.2	0.0262	0.0301
1.4	0.0092	0.0112
1.6	0.0025	0.0043

1. The table shows the forward prices of European-style call options on x at various strike prices, which are shown in the first column.
2. The models using the CRRA and DRRA pricing kernels are calibrated to give the same forward price for the underlying asset. This forward price is shown by the option with zero strike price, in the first row.
3. Prices are computed using a binomial approximation with $n = 10$ and $n = 11$ sub-periods, and taking the average of the two approximations.
4. Prices are computed using the risk-neutral distributions in Table 3.